

Course MA1S11: Michaelmas Term 2016.

Tutorial 9: Sample

December 6–9, 2016

Results that may be useful.

Differentiation by Rule

Let x be a real variable, taking values in a subset D of the real numbers, and let y , u and v dependent variables, expressible as functions of the independent variable x , that are differentiable with respect to x . Then the following results are valid:—

- (i) if $y = c$, where c is a real constant, then $\frac{dy}{dx} = 0$;
- (ii) if $y = cu$, where c is a real constant, then $\frac{dy}{dx} = c \frac{du}{dx}$;
- (iii) if $y = u + v$ then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$;
- (iv) if $y = x^q$, where q is a rational number, then $\frac{dy}{dx} = qx^{q-1}$;
- (v) (*Product Rule*) if $y = uv$ then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$;
- (vi) (*Quotient Rule*) if $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$;
- (vii) (*Chain Rule*) if y is expressible as a differentiable function of u , where u in turn is expressible as a differentiable function of x , then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.

Definitions and Basic Properties of Trigonometric Functions

The *sine function* ($\sin$) sends a real number x to the sine $\sin x$ of an angle measuring x radians. The circumference of a circle of radius one is of length 2π . Radian measure corresponds to distance along the circumference of the unit circle. Therefore four right angles equal 2π radians, and thus one right angle equals $\frac{1}{2}\pi$ radians.

The *cosine function* ($\cos$) satisfies the identity $\cos x = \sin(\frac{1}{2}\pi - x)$ for all real numbers x .

The *tangent function* ($\tan$), *cotangent function* ($\cot$), *secant function* ($\sec$) and *cosecant function* ($\csc$) are defined so that

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

for all real numbers x . These functions satisfy the following identities:—

$$\cos^2 x + \sin^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x,$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y, \quad \cos(x + y) = \cos x \cos y - \sin x \sin y,$$

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - 1 = 1 - 2 \sin^2 x.$$

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) + \cos(x + y)),$$

$$\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y)),$$

$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y)).$$

Derivatives of Trigonometric Functions

The derivatives of the sine, cosine and tangent functions are as follows:—

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x.$$

Derivatives of Logarithm and Exponential Functions

The exponential e^x of x is defined for all real numbers x , and $e^{s+t} = e^s e^t$ for all real numbers t . Moreover

$$\frac{d}{dx}(e^{kx}) = ke^{kx}$$

for all real numbers k . The exponential e^x of x

The natural logarithm function satisfies

$$\frac{d}{dx}(\ln kx) = \frac{1}{x} \quad (x > 0 \text{ and } k > 0)$$

for all positive real numbers k . The natural logarithm $\ln x$ of x is defined for all positive real numbers x , and satisfies $\ln(uv) = \ln u + \ln v$ for all positive real numbers u and v .

Properties of Integrals

Let f and g be integrable functions on a closed bounded interval $[a, b]$, and let c be a real number. Then

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

and

$$\int_a^b (cf(x)) dx = c \int_a^b f(x) dx.$$

Also

$$\int_a^b x^q dx = \frac{1}{q+1}(b^{q+1} - a^{q+1})$$

for all rational numbers q and for all positive real numbers a and b . This identity is also valid for all real numbers a and b in the special case where q is a non-negative integer.

Integrals of sine and cosine are as follows:

$$\int_0^s \sin kx dx = \frac{1}{k}(1 - \cos ks) \quad \text{and} \quad \int_0^s \cos kx dx = \frac{1}{k} \sin ks.$$

Also

$$\int_0^s e^{kx} dx = \frac{1}{k}(e^{ks} - 1).$$

Problem 1

Determine the derivatives of the following functions.

(a)	$e^{\sin(x^2-4x^3)}$	
(b)	$\frac{x^5}{7+2\cos(\ln x)}$	
(c)	$\cos\left(\frac{x^5}{2+(\ln x)^6}\right)$	

Problem 2

Determine the values of the following definite integrals.

(a)	$\int_2^4 (18x^2 - 8x + 6) dx$	
(b)	$\int_0^{\frac{1}{12}\pi} \sin 6x dx$	
(c)	$\int_8^{27} (25x^{\frac{2}{3}} + 12x^{\frac{1}{3}}) dx$	

It is recommended that you show your working on the *Additional Work* sheets attached to the tutorial sheet.

Additional Work

[illegible]

