Course MA1S11: Michaelmas Term 2016.

Tutorial 8: Sample

November 29 to December 2, 2016

Solutions

Results that may be useful.

Quadratic Polynomials

Let a, b and c be real or complex numbers, where $a \neq 0$. The roots of the quadratic polynomial $ax^2 + bx + c$ are given by the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Differentiation by Rule

Let x be a real variable, taking values in a subset D of the real numbers, and let y, u and v dependent variables, expressible as functions of the independent variable x, that are differentiable with respect to x. Then the following results are valid:—

(i) if $y = c$, where c is a real constant, then $\dfrac{dy}{dx} = 0$;

(ii) if $y = cu$, where c is a real constant, then $\dfrac{dy}{dx} = c\dfrac{du}{dx}$;

(iii) if $y = u + v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$;

(iv) if $y = x^q$, where q is a rational number, then $\dfrac{dy}{dx} = qx^{q-1}$;

(v) *(Product Rule)* if $y = uv$ then $\dfrac{dy}{dx} = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$;

(vi) *(Quotient Rule)* if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$;

(vii) *(Chain Rule)* if y is expressible as a differentiable function of u, where u in turn is expressible as a differentiable function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$.

Definitions and Basic Properties of Trigonometric Functions

The *sine function* (sin) sends a real number x to the sine $\sin x$ of an angle measuring x *radians*. The circumference of a circle of radius one is of length 2π. Radian measure corresponds to distance along the circumference of the unit circle. Therefore four right angles equal 2π radians, and thus one right angle equals $\frac{1}{2}\pi$ radians.

The *cosine function* (cos) satisfies the identity $\cos x = \sin(\frac{1}{2}\pi)$ for all real numbers x.

The *tangent function* (tan), *cotangent function* (cot), *secant function* (sec) and *cosecant function* (csc) are defined so that

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

for all real numbers x. These functions satisfy the following identities:—

$$\cos^2 x + \sin^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x,$$

$$\sin(x+y) = \sin x\, \cos y + \cos x\, \sin y, \quad \cos(x+y) = \cos x\, \cos y - \sin x\, \sin y,$$

$$\sin 2x = 2 \sin x\, \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - 1 = 1 - 2\sin^2 x.$$

$$\begin{aligned}
\sin x\, \sin y &= \tfrac{1}{2}(\cos(x-y) + \cos(x+y)), \\
\cos x\, \cos y &= \tfrac{1}{2}(\cos(x+y) + \cos(x-y)), \\
\sin x\, \cos y &= \tfrac{1}{2}(\sin(x+y) + \sin(x-y)).
\end{aligned}$$

Derivatives of Trigonometric Functions

The derivatives of the sine, cosine and tangent functions are as follows:—

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x.$$

Inverse Trigonometric Functions

The *inverse tangent* (arctan), *inverse sine* (arcsin) and *inverse cosine* (arccos) functions are the unique functions

$$\arctan\colon \mathbb{R} \to (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi), \quad \arcsin\colon [-1,1] \to [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi], \quad \arccos\colon [-1,1] \to [0,\pi]$$

for which

$$\begin{aligned}
&\tan(\arctan x) = x \text{ for all } x \in \mathbb{R}, \\
&\sin(\arcsin x) = x \text{ for all } x \in [-1,1], \\
&\cos(\arccos x) = x \text{ for all } x \in [-1,1].
\end{aligned}$$

Marked Problem

Determine the derivatives of the following functions, using standard rules for differentiation such as the Product Rule, Quotient Rule and Chain Rule where appropriate.

(a)	$\cos(13x^3 - 8x^2 + 5)$	$(-39x^2 + 16x)\sin(13x^3 - 8x^2 + 5)$
(b)	$x^7 \sin(8x^5)$	$7x^6 \sin(8x^5) + 40x^{11} \cos(8x^5)$
(c)	$\cos\left(\dfrac{x^4}{x^2 - 6x + 16}\right)$	$\dfrac{-2x^5 + 18x^4 - 64x^3}{(x^2 - 6x + 16)^2} \sin\left(\dfrac{x^4}{x^2 - 6x + 16}\right)$
(d)	$\sin^3(9x^4)$	$108x^3 \sin^2(9x^4)\cos(9x^4)$
(e)	$x^7 \arctan(4x^5)$	$7x^6 \arctan(4x^5) + \dfrac{20x^{11}}{1 + 16x^{10}}$

It is recommended that you show your working on the *Additional Work* sheets attached to the tutorial sheet.

Optional (Unmarked) Problem

(*For those with time on their hands during tutorials*)

The exponential function has not been formally introduced into the MA1S11 lecture notes as yet. It should be familiar from the Leaving Certificate course. It has the following properties:—

$$e^x > 0, \quad e^0 = 1, \quad e^{x+y} = e^x e^y \quad \text{and} \quad \frac{d}{dx}\left(e^{px}\right) = pe^{px}$$

for all real numbers p, x and y.

Show that if the polynomial $r^2 + br + c$ (in the variable r) has two distinct real roots k and m (so that $r^2 + br + c = 0$ when $r = k$ and when $r = m$), and if

$$y = Ae^{kx} + Be^{mx},$$

where A and B are arbitrary real constants, then

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

If time permits, show that if the polynomial $r^2 + br + c$ is of the form $(r - k)^2$ (so that the polynomial has a repeated root at $r = k$), and if

$$y = (Ax + B)e^{kx},$$

where A and B are arbitrary real constants, then

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

If time still permits, show that if the polynomial $r^2 + br + c$ has two non-real roots $p + iq$ and $p - iq$, where p and q are real numbers, $q > 0$ and $i = \sqrt{-1}$, and if

$$y = e^{px}(A\cos qx + B\sin qx),$$

where A and B are arbitrary real constants, then

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$