

# Course MA1S11: Michaelmas Term 2016.

## Tutorial 8: Sample

November 29 to December 2, 2016

Results that may be useful.

### *Quadratic Polynomials*

Let  $a$ ,  $b$  and  $c$  be real or complex numbers, where  $a \neq 0$ . The roots of the quadratic polynomial  $ax^2 + bx + c$  are given by the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

### *Differentiation by Rule*

Let  $x$  be a real variable, taking values in a subset  $D$  of the real numbers, and let  $y$ ,  $u$  and  $v$  dependent variables, expressible as functions of the independent variable  $x$ , that are differentiable with respect to  $x$ . Then the following results are valid:—

- (i) if  $y = c$ , where  $c$  is a real constant, then  $\frac{dy}{dx} = 0$ ;
- (ii) if  $y = cu$ , where  $c$  is a real constant, then  $\frac{dy}{dx} = c \frac{du}{dx}$ ;
- (iii) if  $y = u + v$  then  $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ;
- (iv) if  $y = x^q$ , where  $q$  is a rational number, then  $\frac{dy}{dx} = qx^{q-1}$ ;
- (v) (*Product Rule*) if  $y = uv$  then  $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$ ;
- (vi) (*Quotient Rule*) if  $y = \frac{u}{v}$  then  $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ ;
- (vii) (*Chain Rule*) if  $y$  is expressible as a differentiable function of  $u$ , where  $u$  in turn is expressible as a differentiable function of  $x$ , then  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ .

### Definitions and Basic Properties of Trigonometric Functions

The *sine function* ( $\sin$ ) sends a real number  $x$  to the sine  $\sin x$  of an angle measuring  $x$  radians. The circumference of a circle of radius one is of length  $2\pi$ . Radian measure corresponds to distance along the circumference of the unit circle. Therefore four right angles equal  $2\pi$  radians, and thus one right angle equals  $\frac{1}{2}\pi$  radians.

The *cosine function* ( $\cos$ ) satisfies the identity  $\cos x = \sin(\frac{1}{2}\pi)$  for all real numbers  $x$ .

The *tangent function* ( $\tan$ ), *cotangent function* ( $\cot$ ), *secant function* ( $\sec$ ) and *cosecant function* ( $\csc$ ) are defined so that

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

for all real numbers  $x$ . These functions satisfy the following identities:—

$$\cos^2 x + \sin^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x,$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y, \quad \cos(x + y) = \cos x \cos y - \sin x \sin y,$$

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - 1 = 1 - 2 \sin^2 x.$$

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y)),$$

$$\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y)),$$

$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y)).$$

### Derivatives of Trigonometric Functions

The derivatives of the sine, cosine and tangent functions are as follows:—

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x.$$

### Inverse Trigonometric Functions

The *inverse tangent* ( $\arctan$ ), *inverse sine* ( $\arcsin$ ) and *inverse cosine* ( $\arccos$ ) functions are the unique functions

$$\arctan: \mathbb{R} \rightarrow (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi), \quad \arcsin: [-1, 1] \rightarrow [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi], \quad \arccos: [-1, 1] \rightarrow [0, \pi]$$

for which

$$\tan(\arctan x) = x \text{ for all } x \in \mathbb{R},$$

$$\sin(\arcsin x) = x \text{ for all } x \in [-1, 1],$$

$$\cos(\arccos x) = x \text{ for all } x \in [-1, 1].$$

### Marked Problem

Determine the derivatives of the following functions, using standard rules for differentiation such as the Product Rule, Quotient Rule and Chain Rule where appropriate.

(a)	$\cos(13x^3 - 8x^2 + 5)$	
(b)	$x^7 \sin(8x^5)$	
(c)	$\cos\left(\frac{x^4}{x^2 - 6x + 16}\right)$	
(d)	$\sin^3(9x^4)$	
(e)	$x^7 \arctan(4x^5)$	

It is recommended that you show your working on the *Additional Work* sheets attached to the tutorial sheet.

### Optional (Unmarked) Problem

*(For those with time on their hands during tutorials)*

The exponential function has not been formally introduced into the MA1S11 lecture notes as yet. It should be familiar from the Leaving Certificate course. It has the following properties:—

$$e^x > 0, \quad e^0 = 1, \quad e^{x+y} = e^x e^y \quad \text{and} \quad \frac{d}{dx}(e^{px}) = pe^{px}$$

for all real numbers  $p$ ,  $x$  and  $y$ .

Show that if the polynomial  $r^2 + br + c$  (in the variable  $r$ ) has two distinct real roots  $k$  and  $m$  (so that  $r^2 + br + c = 0$  when  $r = k$  and when  $r = m$ ), and if

$$y = Ae^{kx} + Be^{mx},$$

where  $A$  and  $B$  are arbitrary real constants, then

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

If time permits, show that if the polynomial  $r^2 + br + c$  is of the form  $(r - k)^2$  (so that the polynomial has a repeated root at  $r = k$ ), and if

$$y = (Ax + B)e^{kx},$$

where  $A$  and  $B$  are arbitrary real constants, then

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

If time still permits, show that if the polynomial  $r^2 + br + c$  has two non-real roots  $p + iq$  and  $p - iq$ , where  $p$  and  $q$  are real numbers,  $q > 0$  and  $i = \sqrt{-1}$ , and if

$$y = e^{px}(A \cos qx + B \sin qx),$$

where  $A$  and  $B$  are arbitrary real constants, then

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

## Additional Work

[illegible]





