Course MA1S11: Michaelmas Term 2016.
Tutorial 6: Sample

November 15–18, 2016

Solutions

Results that may be useful.

Quadratic Polynomials

Let a, b and c be real or complex numbers, where $a \neq 0$. The roots of the quadratic polynomial $ax^2 + bx + c$ are given by the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Definition of Continuity

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set of real numbers, and let s be a real number belonging to D. The function f is *continuous* at s if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$.

Definition of Limits

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point belonging to D, and let L be a real number. The real number L is said to be the *limit* of $f(x)$, as x tends to s in D, if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - L| < \varepsilon$ for all real numbers x in D that satisfy $0 < |x - s| < \delta$.

Some Properties of Limits

Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions defined on a subset D of the set of real numbers, and let s be a real number. Suppose that the limits $\lim_{x\to s} f(x)$ and $\lim_{x\to s} g(x)$ exist. Then

$$\begin{aligned}
\lim_{x\to s}(f(x) + g(x)) &= \lim_{x\to s} f(x) + \lim_{x\to s} g(x), \\
\lim_{x\to s}(f(x) - g(x)) &= \lim_{x\to s} f(x) - \lim_{x\to s} g(x), \\
\lim_{x\to s}(f(x) \times g(x)) &= \lim_{x\to s} f(x) \times \lim_{x\to s} g(x).
\end{aligned}$$

If moreover $g(x)$ is non-zero around s, and if $\lim_{x\to s} g(x) \neq 0$, then

$$\lim_{x\to s} \frac{f(x)}{g(x)} = \frac{\lim_{x\to s} f(x)}{\lim_{x\to s} g(x)}.$$

A Specimen ε–δ Proof

Theorem *Let $f\colon D \to \mathbb{R}$ be a continuous real-valued functions defined on a subset D of the set of real numbers, let m and b be real numbers, where $m > 0$, and let $g\colon D \to \mathbb{R}$ be defined so that $g(x) = mf(x) + b$ for all real numbers x belonging to D. Then the function $g\colon D \to \mathbb{R}$ is continuous on D.*

Proof Let s be a real number belonging to D, and let some positive real number ε be given. Let $\varepsilon_0 = \varepsilon/m$. Then $\varepsilon_0 > 0$. The continuity of the function f at s then ensures that there exists some positive real number δ such that

$$f(s) - \varepsilon_0 < f(x) < f(s) + \varepsilon_0$$

whenever $|x - s| < \delta$. Then

$$mf(s) - m\varepsilon_0 < mf(x) < mf(s) + m\varepsilon_0,$$

because $m > 0$, and therefore

$$mf(s) + b - m\varepsilon_0 < mf(x) + b < mf(s) + b + m\varepsilon_0.$$

whenever $|x - s| < \delta$. But $mf(x) + b = g(x)$. and $m\varepsilon_0 = \varepsilon$. We conclude that

$$g(s) - \varepsilon < g(x) < g(s) + \varepsilon$$

whenever $|x - s| < \delta$. Thus the function g is continuous at s. Q.E.D.

Problem 1

Determine the values of the limits below.

(It is recommended that you include intermediate steps, but you should make sure that the final numerical answer is clearly presented, either as a whole number or else as a fraction whose numerator and denominator are whole numbers.)

(a)	$\lim_{x\to 0} \frac{20x^6 + 37x^{13}}{4x^6 + 7x^{18}}$	$= \lim_{x\to 0} \frac{20 + 37x^7}{4 + 7x^{12}} = 5$
(b)	$\lim_{x\to 3} \frac{x^2 + 2x - 15}{x - 3}$	$= \lim_{x\to 3} \frac{(x-3)(x+5)}{x-3} = 8$
(c)	$\lim_{x\to 3} \frac{x^2 + x - 12}{x^2 + 4x - 21}$	$= \lim_{x\to 3} \frac{(x-3)(x+4)}{(x-3)(x+7)} = \lim_{x\to 3} \frac{x+4}{x+7} = \frac{7}{10}$
(d)	$\lim_{x\to 3} \frac{x^3 - 27}{x^3 - 3x^2}$	$= \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x^2(x-3)} = \lim_{x\to 3} \frac{x^2+3x+9}{x^2} = \frac{27}{9} = 3$

Problem 2

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers, and let s and L be real numbers. Suppose that $\lim_{x\to+\infty} f(x) = L$. Prove that $\lim_{u\to 0} f\left(\frac{1}{u^2}\right) = L$.

Solution. *Let a positive real number* ε *be given. Then there exists some positive real number* N *such that* $|f(x) - L| < \varepsilon$ *for all real numbers* x *satisfying* $x > N$*. Let* $\delta = 1/\sqrt{N}$*. If* $0 < |u| < \delta$ *then* $\frac{1}{u^2} > N$*, and therefore* $\left| f\left(\frac{1}{u^2}\right) - L \right| < \varepsilon$*. The result follows. Q.E.D.*