Course MA1S11: Michaelmas Term 2016.

Tutorial 5: Sample

November 1–4, 2016

Solutions

Results that may be useful.

Quadratic Polynomials

Let a, b and c be real or complex numbers, where $a \neq 0$. The roots of the quadratic polynomial $ax^2 + bx + c$ are given by the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Injective, Surjective and Bijective Functions

Let $f: X \to Y$ be a function with *domain* X and *codomain* Y. The *range* $f(X)$ of the function is the subset of the codomain Y defined such that $f(X) = \{f(x) : x \in X\}$.

The function $f: X \to Y$ is *injective* if it maps distinct elements of domain X to distinct elements of the codomain Y. Thus $f: X \to Y$ is injective if and only if

$$u, v \in X \text{ and } f(u) = f(v) \Rightarrow u = v.$$

The function $f: X \to Y$ is *surjective* if $f(X) = Y$. The function $f: X \to Y$ is *bijective* if it is both injective and surjective. An *inverse* $g: Y \to X$ for the function $f: X \to Y$ is a function $g: Y \to X$ with the properties that $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$. A function $f: X \to Y$ has a well-defined inverse $g: Y \to X$ if and only if it is bijective.

Derivatives of Polynomial Functions

A *polynomial* $p(x)$ with *coefficients* $a_0, a_1, \ldots, a_n$ is determined by an equation of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n.$$

where $a_0, a_2, \ldots, a_n$ are real numbers. For polynomials other than the zero polynomial, the *degree* of the polynomial is the largest non-negative integer j for which the coefficient of x^j is non-zero.

The *derivative* $p'(x)$ of the polynomial $p(x)$ satisfies the equation

$$p'(x) = a_1 + 2a_2 x + 3ax^2 + \cdots + na_n x^{n-1}.$$

The *tangent line* to the graph of the function $y = p(x)$ at a point $(s, p(s))$ has the equation

$$y = p(s) + p'(s)(x - s).$$

If polynomial $p(x)$ achieves a local maximum or a local minimum when $x = u$ then $p'(u) = 0$. Thus one can locate potential local minima and maxima by seeking roots of the derivative $p'(x)$.

If I is an interval, and if $p'(x) > 0$ for all $x \in I$ then $p(x)$ is an increasing function of x on I.

If I is an interval, and if $p'(x) < 0$ for all $x \in I$ then $p(x)$ is a decreasing function of x on I.

The *second derivative* $p''(x)$ of $p(x)$ is the derivative of the derivative $p'(x)$ of $p(x)$.

If I is an interval, and if $p'(x)$ is an increasing function of x then the graph of the polynomial $p(x)$ is *concave upwards* on I. Thus if $p''(x) > 0$ for all $x \in I$ then the graph of the polynomial $p(x)$ is concave upwards on I.

If I is an interval, and if $p'(x)$ is a decreasing function of x then the graph of the polynomial $p(x)$ is *concave downwards* on I. Thus if $p''(x) < 0$ for all $x \in I$ then the graph of the polynomial $p(x)$ is concave downwards on I.

A real number x_0 determines a point of inflection $(x_0, p(x_0))$ on the graph $y = p(x)$ of the function $p(x)$ if the graph of that function is concave downwards on one side of the vertical line $x = x_0$ and concave upwards on the other side that line. If a real number x_0 determines a point of inflection on the graph $y = p(x)$ of the function $p(x)$ then $p''(x_0) = 0$.

Intermediate Values of Polynomial Functions

In completing this assignment, the following result may be assumed, *without proof*:—

> *Let a and b be real numbers satisfying $a \leq b$, and let $f\colon [a, b] \to \mathbb{R}$ be a function defined on the closed interval $[a, b]$, where $f(x)$ is expressible as a polynomial in the real variable x throughout $[a, b]$. Let c be a real number that lies between $f(a)$ and $f(b)$. Then there exists a real number s satisfying $a \leq s \leq b$ for which $f(s) = c$.*

Less formally: $f(x)$ passes through all values between $f(a)$ and $f(b)$ as x increases from a to b. (If the function f is neither non-increasing nor non-increasing, it may pass through other values as well.)

It follows from this result that if $f\colon [a, b] \to \mathbb{R}$ is a real-valued function defined on a closed interval $[a, b]$, where $a \leq b$, and if $f(x)$ is expressible as a polynomial in x, then the range $f([a, b])$ of f is the smallest closed interval that contains the values $f(a)$ and $f(b)$ of the function at the endpoints of the interval, together with the values of f at any local minima or maxima that lie within the interval $[a, b]$. These results can be used to determine the range of a polynomial function defined over any interval.

Moreover if a local maximum or minimum falls within the domain of such a function, then the function is not injective.

Problem 1

Let $p(x)$ denote the polynomial defined so that

$$p(x) = x^3 - 3x^2 - 24x + 8.$$

Write down the derivative of this polynomial in the box below:—

$$p'(x) = \boxed{3x^2 - 6x - 24}\,.$$

The derivative $p'(x)$ of the original polynomial $p(x)$ has two real roots x_1 and x_2. Write the values of these roots x_1 and x_2 in the boxes below:—

$$x_1 = \boxed{-2}\,, \quad x_2 = \boxed{4}\,.$$

Write, in the boxes below, the values of the original polynomial $p(x)$, at x_1 and x_2, and, from investigating the behavour of the derivative of $p(x)$, or otherwise, determine whether the function achieves a local minimum or a local minimum at these values, and enter the result in the boxes below:—

the value of $p(x_1)$ is $= \boxed{36}$

and at $x = x_1$ the function achieves a local $\boxed{\text{maximum}}$

the value of $p(x_2)$ is $\boxed{-72}$

and at $x = x_2$ the function achieves a local $\boxed{\text{minimum}}$

(Answer either "minimum" or "maximum" in the relevant boxes above.)

Let $p''(x)$ denote the second derivative of the polynomial $p(x)$ (i.e., the derivative of the derivative of $p(x)$. Write down the second derivative $p''(x)$ of the polynomial $p(x)$ in the box below:—

$$p''(x) = \boxed{6x - 6}\,.$$

The second derivative $p''(x)$ of the original polynomial $p(x)$ has a root at some real number x_0. Write the value of x_0 and $p(x_0)$ in the boxes below:—

If $p''(x_0) = 0$ then $x_0 =$ [1] and $p(x_0) =$ [-18].

The graph of the function has a point of inflexion at $x = x_0$ that is the boundary between the regions where the graph is concave upwards and concave downwards. Describe the qualitative behavour of the function $p(x)$ for $x < x_0$ and $x > x_0$ in the boxes below:

when $x < x_0$ the graph of the function is

[concave downwards];

when $x > x_0$ the graph of the function is

[concave upwards].

(Enter either "concave downwards" or "concave upwards" in the boxes above.)

Determine the equation of the tangent line to the graph $y = p(x)$ of the polynomial function at the point of inflexion $(x_0, p(x_0))$, expressing the equation of the tangent line in the form $y = mx + k$ for appropriate numbers m and k:

Tangent line is [y = 9 - 27x].

Problem 2

This problem continues the study of the polynomial $p(x)$ studied in Problem 1. This polynomial is given by the equation

$$p(x) = x^3 - 3x^2 - 24x + 8.$$

Before attempting this question, you are advised to review the material on *Intermediate Values of Polynomial Functions* included towards the beginning of this tutorial sheet.

The results described there should be used, together with the standard definitions of injective, surjective and bijective functions reproduced on this tutorial sheet, to answer the questions that follow.

***Warning**: if the function f is determined by a polynomial of degree 3 or more, then, except in special cases where some roots are known or can be determined, it is inadvisable to attempt to solve directly equations of the form $f(x) = c$, because in many cases no algebraic formulae in closed form exist for the solutions, and in other cases the algebraic solutions exist but are very complicated, making them effectively unsolvable by techniques taught in school algebra.*

Determine the range of the following functions obtained by restricting the polynomial $x^3 - 3x^2 - 24x + 8$ to closed intervals, and determine whether or not those functions are injective. (Specify the range, and then put a tick under "Is injective" or "not injective", as appropriate.)

(a) The function $f_1\colon [-4, -3] \to \mathbb{R}$ defined such that $f_1(x) = x^3 - 3x^2 - 24x + 8$ for all $x \in [-4, -3]$:

Range	*Is injective*	*Not injective*
[-8,26]	√	

(b) The function $f_2\colon [-3, -2] \to \mathbb{R}$ defined such that $f_1(x) = x^3 - 3x^2 - 24x + 8$ for all $x \in [-3, -2]$:

Range	*Is injective*	*Not injective*
[26,36]	√	

(c) The function $f_3\colon [-2, 1] \to \mathbb{R}$ defined such that $f_1(x) = x^3 - 3x^2 - 24x + 8$ for all $x \in [-2, 1]$:

Range	*Is injective*	*Not injective*
[-18,36]	√	

(d) The function $f_4\colon [-3, 1] \to \mathbb{R}$ defined such that $f_1(x) = x^3 - 3x^2 - 24x + 8$ for all $x \in [-3, 1]$:

Range	*Is injective*	*Not injective*
[-18,36]		√

(e) The function $f_5\colon [1,6] \to \mathbb{R}$ defined such that $f_1(x) = x^3 - 3x^2 - 24x + 8$ for all $x \in [1,6]$:

Range	*Is injective*	*Not injective*
[-72,-18]		√

(f) The function $f_6\colon [-2,4] \to \mathbb{R}$ defined such that $f_1(x) = x^3 - 3x^2 - 24x + 8$ for all $x \in [-2,4]$:

Range	*Is injective*	*Not injective*
[-72,36]	√	