Course MA1S11: Michaelmas Term 2016.

Tutorial 4: Sample

October 25–28, 2016

Solutions

Results that may be useful.

Quadratic Polynomials

Let a, b and c be real or complex numbers, where $a \neq 0$. The roots of the quadratic polynomial $ax^2 + bx + c$ are given by the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Moreover if $a = 1$ then the sum of the roots is $-b$ and the product of the roots is c.

Polynomial Division

The following is a representative example, showing division of the cubic polynomial $x^3 - 9x^2 + 24 - 16$ by the polynomial $x - 3$:—

$$\begin{array}{r|rrrr}
 & x^2 & -6x & +6 & \\ \cline{2-5}
x-3 & x^3 & -9x^2 & +24x & -16 \\
 & x^3 & -3x^2 & & \\ \cline{2-3}
 & & -6x^2 & +24x & \\
 & & -6x^2 & +18x & \\ \cline{3-4}
 & & & 6x & -16 \\
 & & & 6x & -18 \\ \cline{4-5}
 & & & & 2
\end{array}$$

From the above calculation we deduce that

$$x^3 - 9x^2 + 24x - 16 = (x-3)(x^2 - 6x + 6) + 2.$$

Let $p(x)$ be a polynomial. Given any real number s, one can use polynomial division to find a polynomial $q_s(x)$ so that

$$p(x) = (x - s)q_s(x) + p(s).$$

Standard properties of limits then ensure that

$$\lim_{x \to s} \frac{p(x) - p(s)}{x - s} = q_s(s).$$

The value $q_s(s)$ of the polynomial $q_s(x)$ at $x = s$ should therefore be equal to the derivative of the polynomial p at s.

Limits

Let s be a real number, and let D be a subset of the set $\mathbb{R}$ of real numbers. The set D is said to be a *neighbourhood* of s if there exists some positive real number δ such that $(s-\delta, s+\delta) \subset D$, where

$$(s-\delta, s+\delta) = \{x \in \mathbb{R} : s-\delta < x < s+\delta).$$

Definition. Let s and l be real numbers, and let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of $\mathbb{R}$ for which $D \cup \{s\}$ is a neighbourhood of s. We say that l is the *limit* of $f(x)$ as x tends to s, and write

$$\lim_{x \to s} f(x) = l,$$

if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $l - \varepsilon < f(x) < l + \varepsilon$ for all real numbers x in D that satisfy both $s - \delta < x < s + \delta$ and $x \neq s$.

Example

$$\lim_{x \to 1} \frac{3x^2 - 4x + 1}{x - 1} = 2.$$

Indeed let $f\colon \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be defined such that

$$f(x) = \frac{3x^2 - 4x + 1}{x - 1}$$

for all real numbers x satisfying $x \neq 1$. Now $3x^2 - 4x + 1 = (3x-1)(x-1)$ for all real numbers x. It follows that $f(x) = 3x - 1$ whenever $x \neq 1$. Given any strictly positive real number ε, let $\delta = \frac{1}{3}\varepsilon$. If $x \neq 1$ and $1 - \delta < x < 1 + \delta$ then $2 - \varepsilon < f(x) < 2 + \varepsilon$, and thus the definition of limits is satisfied.

Values of Trigonometric Functions

The sine and cosine functions have the following properties:—

$$\sin(\tfrac{1}{2}\pi t) = \sin(\tfrac{1}{2}\pi(t+4)), \quad \cos(\tfrac{1}{2}\pi t) = \cos(\tfrac{1}{2}\pi(t+4)),$$

$$-1 \leq \sin(\tfrac{1}{2}\pi t) \leq 1 \quad \text{and} \quad -1 \leq \cos(\tfrac{1}{2}\pi t) \leq 1$$

for all real numbers t;

$$\sin(\tfrac{1}{2}\pi(4k)) = 0, \quad \cos(\tfrac{1}{2}\pi(4k)) = 1, \quad \sin(\tfrac{1}{2}\pi(4k+1)) = 1,$$

$$\cos(\tfrac{1}{2}\pi(4k+1)) = 0, \quad \sin(\tfrac{1}{2}\pi(4k+2)) = 0, \quad \cos(\tfrac{1}{2}\pi(4k+2)) = -1,$$

$$\sin(\tfrac{1}{2}\pi(4k+3)) = -1 \quad \text{and} \quad \cos(\tfrac{1}{2}\pi(4k+3)) = 0$$

for all integers k.

Problem 1

Let $p(x)$ denote the polynomial defined so that

$$p(x) = x^3 - 10x^2 + 19x + 30.$$

(a) Use polynomial division to find the polynomial $q_4(x)$, where

$$p(x) = (x - 4)q_4(x) + p(4).$$

Enter your answer in the following box:

$$q_4(x) = \boxed{x^2 - 6x - 5}.$$

Calculate the value of this polynomial at 4 and enter it in the following box:

$$q_4(4) = \boxed{-13}.$$

Verify that your answer is equal to the value of the polynomial

$$3x^2 - 20x + 19$$

at $x = 4$.

***Note for solution**: polynomial division can be carried out as follows:*

$$\begin{array}{rrrrr}
 & & x^2 & -\,6x & -\,5 \\ \cline{2-5}
x - 4 \big) & x^3 & -\,10x^2 & +\,19x & +\,30 \\
 & x^3 & -\,4x^2 & & \\ \cline{2-3}
 & & -6x^2 & +\,19x & \\
 & & -6x^2 & +\,24x & \\ \cline{3-4}
 & & & -5x & +\,30 \\
 & & & -5x & +\,20 \\ \cline{4-5}
 & & & & 10
\end{array}$$

(b) Let $s = 7$. Use polynomial division to find the polynomial $q_7(x)$, where

$$p(x) = x^3 - 10x^2 + 19x + 30 = (x - 7)q_7(x) + p(7).$$

Enter your answer in the following box:

$$q_7(x) = \boxed{x^2 - 3x - 2}.$$

Calculate the value of this polynomial at 7 and enter it in the following box:

$$q_7(7) = \boxed{26}.$$

Verify that your answer is equal to the value of the polynomial

$$3x^2 - 20x + 19$$

at $x = 7$.

Note for solution*: polynomial division can be carried out as follows:*

$$\begin{array}{r|rrrr}
 & x^2 & -\,3x & -\,2 & \\
\hline
x-7 & x^3 & -\,10x^2 & +\,19x & +\,30 \\
 & x^3 & -\,7x^2 & & \\
\hline
 & & -3x^2 & +\,19x & \\
 & & -3x^2 & +\,21x & \\
\hline
 & & & -2x & +\,30 \\
 & & & -2x & +\,14 \\
\hline
 & & & & 16
\end{array}$$

Problem 2

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined such that

$$f(x) = \begin{cases} 8x - x^3 & \text{if } x \neq 2; \\ 2 & \text{if } x = 2. \end{cases}$$

Examine carefully the definition of limits (reproduced earlier on this tutorial sheet). Which of the following numbers is equal to $\lim_{x\to 2} f(x)$? (Place a tick in the box below the correct answer.)

1	2	8	12
		$\surd$	

Problem 3

Let $f\colon \mathbb{R} \setminus \{3\} \to \mathbb{R}$ be defined so that

$$f(x) = 5 + 9(x-3)^2 \cos\left(\frac{\pi}{2x-6}\right)$$

for all non-zero real numbers x. In parts (a), (b), (c) below, a formula is given which determines, for each for each real number ε satisfying $0 < \varepsilon < 1$, a positive real number δ_ε corresponding to ε. In each of (a), (b) and (c), determine whether or not it is the case that, for all real numbers ε satisfying $0 < \varepsilon < 1$, the inequalities

$$5 - \varepsilon < 5 + 9(x-3)^2 \cos\left(\frac{\pi}{2x-6}\right) < 5 + \varepsilon$$

are satisfied for all real numbers x satisfying both $x \neq 3$ and

$$3 - \delta_\varepsilon < x < 3 + \delta_\varepsilon.$$

Note for solution: *a straightforwardly workable value of δ_ε should satisfy $9\delta_\varepsilon^2 \leq \varepsilon$, and thus one should ensure that $\delta_\varepsilon \leq \frac{1}{3}\sqrt{\varepsilon}$. And moreover $\frac{1}{3}\varepsilon \leq \frac{1}{3}\sqrt{\varepsilon}$ because $0 < \varepsilon < 1$.*

(a)

$\delta_\varepsilon = \sqrt{\varepsilon}$, | *Yes* | *No* ✓ |

(b)

$\delta_\varepsilon = \frac{1}{3}\sqrt{\varepsilon}$, | *Yes* ✓ | *No* |

(c)

$\delta_\varepsilon = \frac{1}{3}\varepsilon$, | *Yes* ✓ | *No* |