Course MA1S11: Michaelmas Term 2016.

Tutorial 3: Sample

October 18–21, 2016

Solutions

Results that may be useful.

Quadratic Polynomials and related Functions

Let a, b and c be real or complex numbers, where $a \neq 0$. The roots of the quadratic polynomial $ax^2 + bx + c$ are given by the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Moreover if $a = 1$ then the sum of the roots is $-b$ and the product of the roots is c.

Let a and c be real constants, where $a > 0$ and $c > 0$, and let $F\colon (0, +\infty) \to \mathbb{R}$ be the function on the set of positive real numbers defined such that

$$F(x) = ax + \frac{c}{x}$$

for all positive real numbers x. A real number y satisfies the equation $y = F(x)$ for some positive real number x if and only if $ax^2 - yx + c = 0$. The real numbers x (if any) that satisfy the equation $y = F(x)$ can therefore be determined from the quadratic formula. The minimum value of $F(x)$ on $(0, +\infty)$ is $2\sqrt{ac}$, and this value is attained when $x = \sqrt{c/a}$.

Notation for intervals

Let a and b be real numbers, where $a \leq b$. Then

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}, \quad (a, b] = \{x \in \mathbb{R} \mid a < x \leq b\},$$

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}, \quad (a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

$$(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}, \quad (-\infty, a) = \{x \in \mathbb{R} \mid x < a\},$$

$$[a, +\infty) = \{x \in \mathbb{R} \mid x \geq a\}, \quad (a, +\infty) = \{x \in \mathbb{R} \mid x > a\}.$$

Functions

A function $f\colon X \to Y$ maps elements of a set X to elements of a set Y. The set X is the *domain* of the function $f\colon X \to Y$, and the set Y is the *codomain* of this function. The function $f\colon X \to Y$ is *injective* if it maps distinct elements of domain X to distinct elements of the codomain Y. Thus $f\colon X \to Y$ is injective if and only if

$$u, v \in X \text{ and } f(u) = f(v) \Rightarrow u = v.$$

The *range* of the function $f\colon X \to Y$ is the set $f(X)$, where

$$f(X) = \{f(x) | x \in X\}.$$

The function $f\colon X \to Y$ is *surjective* if $f(X) = Y$. The function $f\colon X \to Y$ is *bijective* if it is both injective and surjective. An *inverse* $g\colon Y \to X$ for the function $f\colon X \to Y$ is a function $g\colon Y \to X$ with the properties that $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$. A function $f\colon X \to Y$ has a well-defined inverse $g\colon Y \to X$ if and only if it is bijective.

Problem 1

(a) Let $f\colon (1, 54] \to \mathbb{R}$ be defined such that $f(x) = 3x + \dfrac{108}{x}$ for all $x \in (1, 54]$.

***Note for solution**: note that if $F(x) = 3x + \dfrac{108}{x}$ for all positive real numbers x then $F(x)$ attains a minimum value 18 at $x = 6$. Relevant values are $F(2) = 60$, $F(6) = 36$, $F(18) = 60$, $F(54) = 164$.*

(i) Is the function $f\colon (1, 54] \to \mathbb{R}$ injective? Tick the correct box below:—

Yes	*No* $\surd$

(ii) What is the range of the function $f\colon (1, 54] \to \mathbb{R}$? Write your answer in the box below as an interval in one of the forms $[a, b]$, $[a, b)$, $(a, b]$, (a, b) as appropriate, where a and b are suitably-chosen real numbers satisfying $a < b$.

[36,164]

***Note for solution**: the function $f\colon (1, 54] \to \mathbb{R}$ achieves a minimum value of 36 when $x = 6$, where 6 belongs to the given domain $(1, 54]$; this determines the lower bound of the range; considering also values of the given polynomial at the endpoints of the domain, and noting that the expression defining this function takes the values 60, 36 and 164 at 2, 6 and 54 respectively, we see that the endpoints of the range must be 36 and 164, and that the range itself must be $[36, 164]$. Coincidences such as, for example, $f(2) = f(18)$, show that the function f is not injective.*

(b) Let $g\colon (2,6] \to \mathbb{R}$ be defined such that $g(x) = 3x + \dfrac{108}{x}$ for all $x \in (2,6]$.

(i) Is the function $g\colon (2,6] \to \mathbb{R}$ injective? Tick the correct box below:—

Yes √	*No*

(ii) What is the range of the function $g\colon (2,6] \to \mathbb{R}$? Write your answer in the box below as an interval in one of the forms $[a,b]$, $[a,b)$, $(a,b]$, (a,b) as appropriate, where a and b are suitably-chosen real numbers satisfying $a < b$.

[36,60)

Note for solution*: the function g is decreasing on the given domain. It is therefore injective, and the range is determined by the images of the endpoints of the domain interval under the function g.*

Problem 2

(a) Consider the function $f\colon [0,4] \to [0,64]$ defined such that

$$f(x) = \begin{cases} 37 + 9x & \text{if } 0 \le x \le 3; \\ 64 - x^3 & \text{if } 3 < x \le 4. \end{cases}$$

for all $x \in [0,4]$.

(i) Is the function $f\colon [0,4] \to [0,64]$ injective? Tick the correct box below:—

Yes √	*No*

If you claim that this function is not injective, give a reason for your answer in not more than three sentences in the following box:

N/A

(ii) Is the function $f\colon [0,4] \to [0,64]$ surjective? Tick the correct box below:—

Yes ✓	*No*

If you claim that this function is not surjective, give a reason for your answer in not more than three sentences in the following box:

N/A

(iii) Is the function $f\colon [0,4] \to [0,64]$ bijective? Tick the correct box below:—

Yes ✓	*No*

(b) Consider the function $g\colon [0,4] \to [0,64)$ defined such that

$$g(x) = \begin{cases} 37 + 9x & \text{if } 0 \leq x < 3; \\ 64 - x^3 & \text{if } 3 \leq x \leq 4. \end{cases}$$

for all $x \in [0,4]$.

(i) Is the function $g\colon [0,4] \to [0,64)$ injective? Tick the correct box below:—

Yes	*No* ✓

If you claim that this function is not injective, give a reason for your answer in not more than three sentences in the following box:

The function $g\colon [0,3] \to [0,27)$ satisfies $g(0) = 37 = g(3)$, and therefore this function is not injective.

(ii) Is the function $g\colon [0,4] \to [0,64)$ surjective? Tick the correct box below:—

Yes $\surd$	*No*

If you claim that this function is not surjective, give a reason for your answer in not more than three sentences in the following box:

N/A

(iii) Is the function $g\colon [0,4] \to [0,64)$ bijective? Tick the correct box below:—

Yes $\surd$	*No*

(c) Consider the function $h\colon [0,4] \to [0,64]$ defined such that

$$h(x) = \begin{cases} 64 - 8x & \text{if } 0 \leq x < 3; \\ x^3 - 27 & \text{if } 3 \leq x \leq 4. \end{cases}$$

for all $x \in [0,4]$.

(i) Is the function $h\colon [0,4] \to [0,64]$ injective? Tick the correct box below:—

Yes √	*No*

If you claim that this function is not injective, give a reason for your answer in not more than three sentences in the following box:

N/A

(ii) Is the function $h\colon [0,4] \to [0,64]$ surjective? Tick the correct box below:—

Yes	*No* √

If you claim that this function is not surjective, give a reason for your answer in not more than three sentences in the following box:

The range of this function does not include real numbers in the interval (37,40].

(iii) Is the function $h\colon [0,4] \to [0,64]$ bijective? Tick the correct box below:—

Yes	*No* √