Course MA1S11: Michaelmas Term 2016.
Tutorial 10: Group Sample
December 13–16, 2016

Results that may be useful.

Differentiation by Rule

Let x be a real variable, taking values in a subset D of the real numbers, and let y, u and v dependent variables, expressible as functions of the independent variable x, that are differentiable with respect to x. Then the following results are valid:—

(i) if $y = c$, where c is a real constant, then $\dfrac{dy}{dx} = 0$;

(ii) if $y = cu$, where c is a real constant, then $\dfrac{dy}{dx} = c\dfrac{du}{dx}$;

(iii) if $y = u + v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$;

(iv) if $y = x^q$, where q is a rational number, then $\dfrac{dy}{dx} = qx^{q-1}$;

(v) *(Product Rule)* if $y = uv$ then $\dfrac{dy}{dx} = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$;

(vi) *(Quotient Rule)* if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$;

(vii) *(Chain Rule)* if y is expressible as a differentiable function of u, where u in turn is expressible as a differentiable function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$.

Definitions and Basic Properties of Trigonometric Functions

The *sine function* (sin) sends a real number x to the sine $\sin x$ of an angle measuring x *radians*. The circumference of a circle of radius one is of length 2π. Radian measure corresponds to distance along the circumference of the unit circle. Therefore four right angles equal 2π radians, and thus one right angle equals $\frac{1}{2}\pi$ radians.

The *cosine function* (cos) satisfies the identity $\cos x = \sin(\frac{1}{2}\pi - x)$ for all real numbers x.

The *tangent function* (tan), *cotangent function* (cot), *secant function* (sec) and *cosecant function* (csc) are defined so that

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

for all real numbers x. These functions satisfy the following identities:—

$$\cos^2 x + \sin^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x,$$

$$\sin(x+y) = \sin x\, \cos y + \cos x\, \sin y, \quad \cos(x+y) = \cos x\, \cos y - \sin x\, \sin y,$$

$$\sin 2x = 2\sin x\, \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - 1 = 1 - 2\sin^2 x.$$

$$\begin{aligned} \sin x\, \sin y &= \tfrac{1}{2}(\cos(x-y) + \cos(x+y)), \\ \cos x\, \cos y &= \tfrac{1}{2}(\cos(x+y) + \cos(x-y)), \\ \sin x\, \cos y &= \tfrac{1}{2}(\sin(x+y) + \sin(x-y)). \end{aligned}$$

Derivatives of Trigonometric Functions

The derivatives of the sine, cosine and tangent functions are as follows:—

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x.$$

Derivatives of Logarithm and Exponential Functions

The exponential e^x of x is defined for all real numbers x, and $e^{s+t} = e^s e^t$ for all real numbers t. Moreover

$$\frac{d}{dx}(e^{kx}) = ke^{kx}$$

for all real numbers k. The exponential e^x of x

The natural logarithm function satisfies

$$\frac{d}{dx}(\ln kx) = \frac{1}{x} \quad (x > 0 \text{ and } k > 0)$$

for all positive real numbers k. The natural logarithm $\ln x$ of x is defined for all positive real numbers x, and satisfies $\ln(uv) = \ln u + \ln v$ for all positive real numbers u and v.

Properties of Integrals

Let f and g be integrable functions on a closed bounded interval $[a, b]$, and let c be a real number. Then

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and

$$\int_a^b (cf(x))\,dx = c\int_a^b f(x)\,dx.$$

Also

$$\int_a^b x^q\,dx = \frac{1}{q+1}(b^{q+1} - a^{q+1})$$

for all rational numbers q and for all positive real numbers a and b. This identity is also valid for all real numbers a and b in the special case where q is a non-negative integer.

Integrals of sine and cosine are as follows:

$$\int_0^s \sin kx\,dx = \frac{1}{k}(1 - \cos ks) \quad \text{and} \quad \int_0^s \cos kx = \frac{1}{k}\sin ks.$$

Also

$$\int_1^s \frac{1}{x}\,dx = \ln s \qquad (s > 0),$$

$$\int_0^s e^{kx}\,dx = \frac{1}{k}(e^{ks} - 1).$$

Integration by Substitution

Let x be a real variable taking values in a closed interval $[a, b]$, and let $u = \varphi(x)$ for all $x \in [a, b]$, where $\varphi\colon [a, b] \to \mathbb{R}$ be a continuously-differentiable function on the interval $[a, b]$. The rule for Integration by Substitution can then be stated as follows:

$$\int_{u(a)}^{u(b)} f(u)\,du = \int_a^b f(u(x)))\,\frac{du}{dx}\,dx.$$

for all continuous real-valued functions f whose domain includes $u(x)$ for all real numbers x satisfying $a \leq x \leq b$, where $u(a)$ and $u(b)$ denote the values of u when $x = a$ and $x = b$ respectively.

Problems

Evaluate the following integrals using the method of Integration by Substitution:—

(a)	$\displaystyle\int_1^3 \frac{x^5}{\left(3x^6+2\right)^2}\,dx$	$\dfrac{364}{32835}$
(b)	$\displaystyle\int_0^s x^6 e^{-4x^7}\,dx$	$\dfrac{1}{28}-\dfrac{1}{28}e^{-4s^7}$
(c)	$\displaystyle\int_0^s \frac{x}{5x^2+1}\,dx$	$\dfrac{1}{10}\log\left(5s^2+1\right)$
(d)	$\displaystyle\int_0^s -\frac{x^3}{2}\sin^2\left(x^4\right)\cos\left(x^4\right)dx$	$-\dfrac{1}{24}\sin^3\left(s^4\right)$

It is recommended that you show your working on the *Additional Work* sheets attached to the tutorial sheet.

Detailed solutions to problems:

(a)

$$\begin{aligned}\int_1^3 \frac{x^5}{\left(3x^6+2\right)^2}\,dx &= \left[-\frac{1}{54x^6+36}\right]_1^3 \\ &= \left(-\frac{1}{39402}\right)-\left(-\frac{1}{90}\right) \\ &= \frac{364}{32835}\end{aligned}$$

(b)

$$\begin{aligned}\int_0^s x^6 e^{-4x^7}\,dx &= \left[-\frac{1}{28}e^{-4x^7}\right]_0^s \\ &= \left(-\frac{1}{28}e^{-4s^7}\right)-\left(-\frac{1}{28}\right) \\ &= \frac{1}{28}-\frac{1}{28}e^{-4s^7}\end{aligned}$$

(c)

$$\begin{aligned}\int_0^s \frac{x}{5x^2+1}\,dx &= \left[\frac{1}{10}\log\left(5x^2+1\right)\right]_0^s \\ &= \left(\frac{1}{10}\log\left(5s^2+1\right)\right)-(0) \\ &= \frac{1}{10}\log\left(5s^2+1\right)\end{aligned}$$

(d)

$$\begin{aligned}\int_0^s -\frac{x^3}{2}\sin^2\left(x^4\right)\cos\left(x^4\right)dx &= \left[-\frac{1}{24}\sin^3\left(x^4\right)\right]_0^s \\ &= \left(-\frac{1}{24}\sin^3\left(s^4\right)\right)-(0) \\ &= -\frac{1}{24}\sin^3\left(s^4\right)\end{aligned}$$