MA1S11—Calculus Portion
School of Mathematics, Trinity College
Michaelmas Term 2016
Lecture 30 (December 13, 2016)

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Lemma 9.8

Let P be a point lying on an ellipse

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1 - e^2}$, and let O denote the origin $(0, 0)$ of the Cartesian coordinate system. Then the distance $|OP|$ between the points O and P satisfies

$$|OP| = a(1 - e^2) - ex.$$

Proof

The equation of the ellipse ensures that

$$\begin{aligned} y^2 &= a^2(1-e^2) - (1-e^2)(x+ae)^2 \\ &= a^2(1-e^2)^2 - 2ae(1-e^2)x - (1-e^2)x^2 \end{aligned}$$

It follows that

$$\begin{aligned} |OP|^2 &= x^2 + y^2 = e^2x^2 - 2ae(1-e^2)x + a^2(1-e^2)^2 \\ &= (a(1-e^2) - ex)^2 \end{aligned}$$

The result therefore follows on taking square roots. ∎

Lemma 9.9

Let P be a point lying on an ellipse

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$, let O denote the origin $(0,0)$ of the Cartesian coordinate system and let $G = (-2ae, 0)$. Then the distances $|OP|$ and $|GP|$ of the point P from the points O and G respectively satisfy the equation

$$|OP| + |GP| = 2a.$$

Proof

The ellipse is invariant under the reflection of the plane in the line $x = -ae$ that sends the point (x, y) to the point $(-x - 2ae, y)$, because

$$\frac{((-x - 2ae) + ae)^2}{a^2} + \frac{y^2}{b^2} = \frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2}.$$

This reflection preserves distances and swaps the points O and G. It follows from Lemma 9.8 that

$$|GP| = a(1 - e^2) - e(-x - 2ae) = a(1 + e^2) + ex,$$

and therefore

$$|OP| + |GP| = a(1 - e^2) - ex + a(1 + e^2) + ex = 2a,$$

as required. ∎

An ellipse in a plane that is not a circle is determined by two distinct points F and G of that plane, together with a positive real number a, and consists of those points P of the plane for which $|FP| + |GP| = 2a$. These points F and G are referred to as the *foci* of the ellipse. Lemma 9.9 ensures that if the ellipse is specified by an equation of the form

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$, then the points O and G with Cartesian coordinates $(0, 0)$ and $(-2ae, 0)$ respectively are the foci of the ellipse. We deduce from this the following theorem.

Theorem 9.10

Let a particle move in a closed orbit in the plane so that its acceleration is always directed towards the origin and is inversely proportional to the square of the distance from the origin. Then the orbit of the particle is an ellipse, and the origin is located at one of the two foci of that ellipse.

We now discuss the case where eccentricity e of the orbit of the particle is greater satisfies $e > 1$. We have already shown that the equation

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

is satisfied along the orbit of the particle. The points on the orbit of the particle therefore satisfy the equstion

$$\frac{(x + ae)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b = a\sqrt{e^2 - 1}$. This is the equation of a *hyperbola*. The hyperbola however has two "branches" which are separate pieces disconnected from one another. One "branch" consists of those points of the hyperbola for which $a + ex \geq a$, and the other "branch" consists of those points of the hyperbola for which $a + ex \leq -a$.

Lemma 9.11

Let P be a point lying on an hyperbola

$$\frac{(x+ae)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $e > 1$ and $b = a\sqrt{e^2 - 1}$, and let O denote the origin $(0, 0)$ of the Cartesian coordinate system. Then the distance $|OP|$ between the points O and P satisfies

$$|OP| = |a(e^2 - 1) + ex|,$$

and therefore $|OP| = a(e^2 - 1) + ex$ on the branch of the hyperbola for which $x \geq -(e - 1)a$, and $|OP| = -a(e^2 - 1) - ex$ on the branch of the hyperbola for which $x \leq -(e + 1)a$.

Proof

The equation of the hyperbola ensures that

$$\begin{aligned} y^2 &= (e^2-1)(x+ae)^2 - a^2(e^2-1) \\ &= a^2(e^2-1)^2 + 2ae(e^2-1)x + (e^2-1)x^2 \end{aligned}$$

It follows that

$$\begin{aligned} |OP|^2 &= x^2+y^2 = e^2x^2 + 2ae(e^2-1)x + a^2(e^2-1)^2 \\ &= (a(e^2-1)+ex)^2 \end{aligned}$$

The result therefore follows on taking square roots. ∎

Lemma 9.12

Let P be a point lying on a hyperbola

$$\frac{(x+ae)^2}{a^2}+\frac{y^2}{b^2}=1,$$

where the real numbers a, b and e satisfy $a>0$, $0\le e<1$ and $b=a\sqrt{1-e^2}$, let O denote the origin $(0,0)$ of the Cartesian coordinate system and let $G=(-2ae,0)$. Then the distances $|OP|$ and $|GP|$ of the point P from the points O and G respectively satisfy the equation $|GP|-|OP|=2a$ on the branch of the hyperbola on which $x\ge -(e-1)a$, and satisfy the equation $|OP|-|GP|=2a$ on the branch of the hyperbola on which $x\le -(e+1)a$.

Proof

It follows from Lemma 9.11 that if $x \geq (1-e)a$ then $|OP| = a(e^2-1) + ex$, and if P is on that branch of the hyperbola for which $x \leq -(1+e)a$ then $|OP| = -a(e^2-1) - ex$. The transformation of the plane which sends (x, y) to $(-x - 2ae, y)$ interchanges the points O and G, where $G = (-2ae, 0)$, and also interchanges the two branches of the hyperbola. It follows that if P lies on the branch of the hyperbola for which $x \geq (1-e)a$ then

$$|GP| = -a(e^2-1) + ex + 2ae^2 = a(e^2+1) + ex$$

It follows that

$$|GP| - |OP| = 2a.$$

The corresponding result on the other branch of the hyperbola can then be deduced on making use of the transformation sending (x, y) to $(-x - 2ae, y)$ which preserves distances and swaps the two branches of the hyperbola. ∎

Let

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

be the equation of a hyperbola, where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1 - e^2}$, let O denote the origin $(0, 0)$ of the Cartesian coordinate system and let $G = (-2ae, 0)$. The points O and G are referred to as the *foci* of the hyperbola. We associate the focus O with the branch of the hyperbola on which $x \geq -(e - 1)a$, and refer to it as the *primary focus* corresponding to that branch of the hyperbola. Every line passing through the primary focus intersects the corresponding branch of the hyperbola, but lines passing through the other focus do not necessarily intersect this branch. It follows from Lemma 9.12 that the branch of the hyperbola which has O as its primary focus is the locus of points P whose distances $|OP|$ and $|GP$ from the foci O and G satisfy the equation $|GP| - |OP| = 2a$.

Suppose that a particle moves under the influence of an attractive force directed towards the point O and having magnitude μ/r^2, where μ is a constant. Then the equation determining the motion of the particle can be expressed in the form

$$r^2 \frac{d\theta}{dt} = h \quad \text{and} \quad \frac{\ell}{r} = 1 + e\cos\theta,$$

where θ denotes the angle in radians that the line joining the point O to the particle makes with some suitably chosen fixed direction, and where e, h and ℓ are constants that satisfy the conditions $e \geq 0$ and $\ell = h^2/\mu$ (see Theorem 9.6). The quantity e determines the shape of the orbit or trajectory: the orbit is a circle if $e = 0$; the orbit is an ellipse if $0 \leq e < 1$, and the point O is situated at one of the foci of the ellipse; the trajectory is a parabola if $e = 1$; the trajectory is a branch of a hyperbola if $e > 1$, and the point O is located at the primary focus of that branch of the hyperbola.

We now consider the rate at which the area swept out by a line segment joining the fixed point O to the particle increases with time. The Cartesian components of the position of the particle at time t are $x(t)$ and $y(t)$, where

$$x(t) = r(t)\cos\theta(t) \quad \text{and} \quad y(t) = r(t)\sin\theta(t).$$

Let $P(t)$ denote the position of the particle at time t, It follows from linear algebra that, given a small increment Δt of time, the triangle with vertices O, $P(t)$ and $P(t+\Delta t)$ has area $\alpha_t(\Delta t)$, where

$$
\begin{aligned}
\alpha_t(\Delta t) &= \frac{1}{2}\left|\begin{array}{cc} r(t)\,\cos\theta(t) & r(t+\Delta t)\,\cos\theta(t+\Delta t) \\ r(t)\,\sin\theta(t) & r(t+\Delta t)\,\sin\theta(t+\Delta t) \end{array}\right| \\
&= \frac{1}{2}r(t)r(t+\Delta t)\left|\begin{array}{cc} \cos\theta(t) & \cos\theta(t+\Delta t) \\ \sin\theta(t) & \sin\theta(t+\Delta t) \end{array}\right| \\
&= \frac{1}{2}r(t)r(t+\Delta t)\Big(\cos\theta(t)\sin\theta(t+\Delta t) \\
&\qquad\qquad - \sin\theta(t)\cos\theta(t+\Delta t)\Big) \\
&= \frac{1}{2}r(t)r(t+\Delta t)\sin\Big(\theta(t+\Delta t) - \theta(t)\Big).
\end{aligned}
$$

It follows that

$$\begin{aligned}
\lim_{\Delta t \to 0} \frac{\alpha_t(\Delta t)}{\Delta t} &= \tfrac{1}{2} r(t) \left(\lim_{\Delta t \to 0} r(t + \Delta t) \right) \\
&\qquad \times \left(\lim_{\Delta t \to 0} \frac{\sin\Bigl(\theta(t + \Delta t) - \theta(t)\Bigr)}{\Delta t} \right) \\
&= \tfrac{1}{2} (r(t))^2 \, \frac{d}{d\tau} \sin(\theta(t + \tau) - \theta(t)) \Big|_{\tau = 0} \\
&= \tfrac{1}{2} (r(t))^2 \cos(0) \, \frac{d}{d\tau} (\theta(t + \tau) - \theta(t)) \Big|_{\tau = 0} \\
&= \tfrac{1}{2} (r(t))^2 \frac{d\theta(t)}{dt}
\end{aligned}$$

We now assume that if $A(t)$ represents the area swept out by the line joining the point O to the particle since some fixed time, and thus if $A(t+\Delta) - A(t)$ represents the increment to that area over the time interval from t to $t + \Delta t$, then

$$\lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\alpha_t(\Delta))} = 1.$$

(This will be the case if, given any positive real number ε, the trajectory of the particle lies between lines parallel to that passing through the points $P(t)$ and $P(t+\Delta t)$ and lying a distance no more than ε on either side of the line through $P(t)$ and $P(t+\Delta t)$.)

We conclude that

$$\begin{aligned}\frac{dA(t)}{dt} &= \left(\lim_{\Delta t\to 0}\frac{A(t+\delta t)-A(t)}{\alpha_t(\Delta t)}\right)\times\left(\lim_{\Delta t\to 0}\frac{\alpha_t(\Delta t)}{\Delta t}\right)\\ &= \tfrac{1}{2}(r(t))^2\,\frac{d\theta(t)}{dt}.\end{aligned}$$

Now, for a particle whose acceleration is directed always towards the point O, the functions $r(t)$ and $\theta(t)$ determining its trajectory satisfy the equation

$$r(t)^2\frac{d\theta(t)}{dt} = h$$

for some constant h (see Corollary 9.3). It follows that

$$\frac{dA(t)}{dt} = \tfrac{1}{2}h.$$

We have therefore proved the following theorem.

Theorem 9.13

Suppose that a particle moves in the plane so as to ensure that its acceleration is always directed towards a fixed point O. Then the line segment joining O to the particle sweeps out equal areas in equal times.

We now return to the case where the acceleration of the particle towards the fixed point O is equal to μ/r^2, where μ is a constant, and r is the distance of the particle from the point O. We suppose that the particle moves in a fixed elliptic orbit with equation

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$.

The area of the ellipse is then πab. Indeed the ellipse can be obtained from the unit circle by stretching by a factor a in the x-direction, stretching by a factor b in the y-direction, and then translating the centre from $(0,0)$ to $(-ae,0)$. Therefore the area of the ellipse is the area π of the unit circle successively multiplied by a and b.
It follows that the *period* T of the elliptical motion (i.e., the time taken to complete a circle is given by the equation

$$T = \frac{2\pi ab}{h} = \frac{2\pi a^2\sqrt{1-e^2}}{h}.$$

Moreover

$$\frac{h^2}{\mu} = \ell = a(1-e^2)$$

(compare Theorem 9.6 and Corollary 9.7). It follows that

$$T^2 = \frac{4\pi^2 a^4(1-e^2)}{h^2} = \frac{4\pi^2 a^4(1-e^2)}{\mu a(1-e^2)} = \frac{4\pi^2 a^3}{\mu}.$$

We have therefore proved the following theorem.

Theorem 9.14

Let a particle in the plane move in an elliptical orbit where the acceleration of the particle is directed towards one of the foci of that ellipse and has magnitude μ/r^2, where r is the distance of the particle from that focus. Then the period of the motion satisfies the equation

$$T = 2\pi\sqrt{\frac{a^3}{\mu}},$$

where a is the semi-major axis of the ellipse.