MA1S11—Calculus Portion
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9.1. Motion under a Central Force

Let the position of a particle in the plane be determined by two variables $r(t)$ and $\theta(t)$ that are functions of time that may be repeatedly differentiated any number of times, and that determine the Cartesian coordinates $(x(t), y(t))$ of the particle at time t according to the equations

$$x(t) = r(t) \cos\theta(t), \quad y(t) = r(t) \sin\theta(t).$$

Thus $r(t)$ represents the distance of the particle from the origin at time t, and $\theta(t)$ denotes the angle in radians that the line joining the particle to the origin makes with the x-axis at time t.

We denote the first and second derivatives of the quantities $r(t)$, $\theta(t)$, $x(t)$, $y(t)$ by putting dots over the respective letters so that

$$\dot{r}(t) = \frac{dr(t)}{dt}, \quad \dot{\theta}(t) = \frac{d\theta(t)}{dt},$$

$$\dot{x}(t) = \frac{dx(t)}{dt}, \quad \dot{y}(t) = \frac{dy(t)}{dt},$$

$$\ddot{r}(t) = \frac{d^2 r(t)}{dt^2}, \quad \ddot{\theta}(t) = \frac{d^2\theta(t)}{dt^2},$$

$$\ddot{x}(t) = \frac{d^2 x(t)}{dt^2}, \quad \ddot{y}(t) = \frac{d^2 y(t)}{dt^2}.$$

Now $\dot{x}(t)$ and $\dot{y}(t)$ are the Cartesian components of the velocity of the particle at time t. Differentiating we find that

$$\begin{aligned}
\dot{x}(t) = \frac{dx(t)}{dt} &= \frac{d}{dt}\Big(r(t)\cos\theta(t)\Big) \\
&= \frac{dr(t)}{dt}\,\cos\theta(t) - r(t)\frac{d(\theta(t))}{dt}\,\sin\theta(t) \\
&= \dot{r}(t)\,\cos\theta(t) - r(t)\dot{\theta}(t)\,\sin\theta(t), \\
\dot{y}(t) = \frac{dy(t)}{dt} &= \frac{d}{dt}\Big(r(t)\sin\theta(t)\Big) \\
&= \frac{dr(t)}{dt}\,\sin\theta(t) + r(t)\frac{d(\theta(t))}{dt}\,\cos\theta(t) \\
&= \dot{r}(t)\,\sin\theta(t) + r(t)\dot{\theta}(t)\,\cos\theta(t).
\end{aligned}$$

It is not necessary to indicate explicitly the time dependence of the quantities r, θ, x, y, $\dot{r}$, $\dot{\theta}$ $\dot{x}$ and $\dot{y}$. We may therefore simplify notation by writing the equations just derived in the form

$$\begin{aligned} \dot{x} = \frac{dx}{dt} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\ \dot{y} = \frac{dy}{dt} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \end{aligned}$$

Lemma 9.1

Let a particle move in the plane so that the Cartesian components x and y of its position satisfy

$$x = r\cos\theta \quad \textit{and} \quad y = r\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Then

$$\begin{aligned} \frac{dx}{dt} &= \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta, \\ \frac{dy}{dt} &= \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta. \end{aligned}$$

We differentiate again to find the Cartesian components $\ddot{x}(t)$ and $\ddot{y}(t)$ of the acceleration of the particle in terms of $r(t)$, $\theta(t)$ and their first and second derivatives. We find that

$$\begin{aligned}
\ddot{x} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) \\
&= \frac{d}{dt}\left(\dot{r}\cos\theta\right) - \frac{d}{dt}\left(r\dot{\theta}\sin\theta\right) \\
&= \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta \\
&\qquad - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\sin\theta
\end{aligned}$$

Similarly

$$\begin{aligned}
\ddot{y} &= \frac{d}{dt}\left(\frac{dy}{dt}\right) \\
&= \frac{d}{dt}\left(\dot{r}\sin\theta\right) + \frac{d}{dt}\left(r\dot{\theta}\cos\theta\right) \\
&= \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta \\
&\qquad + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos\theta
\end{aligned}$$

We now suppose that the acceleration of the particle is always directed towards or away from the origin, and that its magnitude is determined by the distance of the particle from the origin. Thus we suppose that, when the particle is located at a distance r from the origin, its acceleration has magnitude $|g(r)|$, where $g(r)$ is a function of r defined for positive real numbers r, and is directed towards the origin when $g(r) > 0$, and away from the origin when $g(r) < 0$. Then

$$\ddot{x}(t) = -g(r(t))\cos\theta(t), \quad \ddot{y}(t) = -g(r(t))\sin\theta(t),$$

Simplifying notation by suppressing explicit reference to the time-dependence of the quantities involved, we find that

$$\begin{aligned} -g(r)\cos\theta = \ddot{x} &= \left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\sin\theta, \\ -g(r)\sin\theta = \ddot{y} &= \left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos\theta. \end{aligned}$$

It follows that

$$\begin{aligned}
0 &= \ddot{x}\sin\theta - \ddot{y}\cos\theta = -\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)(\sin^2\theta + \cos^2\theta) \\
&= -\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right), \\
-g(r) &= \ddot{x}\cos\theta + \ddot{y}\sin\theta \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)(\sin^2\theta + \cos^2\theta) \\
&= \ddot{r} - r\dot{\theta}^2.
\end{aligned}$$

We conclude therefore that

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad \text{and} \quad \ddot{r} - r\dot{\theta}^2 = -g(r).$$

We summarize the some of the main results obtained so far in the following proposition.

Proposition 9.2

Let a particle move in the plane so that the Cartesian components x and y of its position satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that the acceleration of the particle satisfies the equations

$$\frac{d^2x}{dt^2} = -g(r)\cos\theta \quad \text{and} \quad \frac{d^2y}{dt^2} = -g(r)\sin\theta$$

at all times, where $g(r)$ is a function of r. Then

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0 \quad \text{and} \quad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -g(r).$$

Corollary 9.3

Let a particle move in the plane so that the Cartesian components x and y of its position satisfy

$$x = r\cos\theta \quad \textit{and} \quad y = r\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that the acceleration of the particle satisfies the equations

$$\frac{d^2x}{dt^2} = -g(r)\cos\theta \quad \textit{and} \quad \frac{d^2y}{dt^2} = -g(r)\sin\theta$$

at all times, where $g(r)$ is a function of r. Then

$$r^2\frac{d\theta}{dt} = h \quad \textit{and} \quad \frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -g(r),$$

where h is a constant (having the same value at all times).

Proof

Differentiating, and applying the result of Proposition 9.2

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2} = r\Bigl(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\Bigr) = 0.$$

It follows that

$$r^2\frac{d\theta}{dt} = h$$

where h is a constant. (In particular the value of h is fixed for all times.) It follows from Proposition 9.2 that

$$-g(r) = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2r}{dt^2} - \frac{h^2}{r^3},$$

as required. ∎

There is a standard method for solving equations of motion for particle moving in the plane, attributed to Jacques Philippe Marie Binet (1786–1856), where the acceleration is directed towards a fixed point located at the origin of a Cartesian coordinate system, and is determined by the distance of the particle from that fixed point, which has been handed down the generations. We suppose that the distance r of the particle from the origin is expressible as a function of the angle θ, at least over sufficiently short periods of time, and set

$$r(t) = \frac{1}{u(\theta(t))},$$

where $u(\theta)$ is a function of θ whose values are positive. Then

$$
\begin{aligned}
\frac{dr}{dt} &= -\frac{1}{(u(\theta))^2}\,\frac{d}{dt}\Big(u(\theta(t))\Big) \\
&= -r^2\,\frac{du}{d\theta}\,\frac{d\theta}{dt} \\
&= -h\frac{du}{d\theta},
\end{aligned}
$$

where $h = r^2\dot{\theta}$. Now h is a constant whose value is the same at all times (Corollary 9.3). Thus if we differentiate again, we find that

$$
\begin{aligned}
\frac{d^2 r}{dt^2} &= -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\,\frac{d^2 u}{d\theta^2}\,\frac{d\theta}{dt} \\
&= -\frac{h^2}{r^2}\frac{d^2 u}{d\theta^2} = -h^2 u^2\,\frac{d^2 u}{d\theta^2}.
\end{aligned}
$$

It then follows from Corollary 9.3 that

$$-g\left(\frac{1}{u}\right) = -h^2 u^2 \frac{d^2 u}{d\theta^2} - h^2 u^3.$$

It follows that

$$\frac{d^2 u}{d\theta^2} + u = \frac{1}{h^2 u^2} g\left(\frac{1}{u}\right).$$

This equation is referred to as the *Binet equation*.
We summarize the result just obtained in a proposition.

Proposition 9.4

Let a particle move in the plane so that the Cartesian components x and y of its position satisfy

$$x = r\,\cos\theta \quad \text{and} \quad y = r\,\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that

$$\frac{d^2x}{dt^2} = -g(r)\cos\theta \quad \text{and} \quad \frac{d^2y}{dt^2} = -g(r)\sin\theta$$

at all times, where $g(r)$ is a positive function of r. Then there exists a constant h such that

$$\frac{d\theta}{dt} = hu^2 \quad \text{and} \quad \frac{d^2u}{d\theta^2} + u = \frac{1}{h^2u^2}\,g\left(\frac{1}{u}\right),$$

where $u = 1/r$.

Corollary 9.5

Let a particle move in the plane so that the Cartesian components x and y of its position satisfy

$$x = r\,\cos\theta \quad \textit{and} \quad y = r\,\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that

$$\frac{d^2x}{dt^2} = -\frac{\mu}{r^2}\cos\theta \quad \textit{and} \quad \frac{d^2y}{dt^2} = -\frac{\mu}{r^2}\sin\theta$$

at all times, where μ is a positive constant. Then there exists a constant h such that

$$\frac{d\theta}{dt} = hu^2 \quad \textit{and} \quad \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

where $u = 1/r$.

Theorem 9.6

Let a particle move in the plane so that the Cartesian components x and y of its position satisfy

$$x = r\,\cos\theta \quad \textit{and} \quad y = r\,\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that

$$\frac{d^2x}{dt^2} = -\frac{\mu}{r^2}\cos\theta \quad \textit{and} \quad \frac{d^2y}{dt^2} = -\frac{\mu}{r^2}\sin\theta$$

at all times, where μ is a positive constant. Then there exist constants h, θ_0 and e, where $e \geq 0$, such that

$$r^2\frac{d\theta}{dt} = h \quad \textit{and} \quad \frac{h^2}{\mu r} = 1 + e\,\cos(\theta - \theta_0).$$

Proof

It follows from Corollary 9.5 that there exists a constant h such that

$$\frac{d\theta}{dt} = hu^2 \quad \text{and} \quad \frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2},$$

where $u = 1/r$. Let

$$w = \frac{1}{\cos\theta}\left(u - \frac{\mu}{h^2}\right)$$

throughout the time period over which the equations of motion of the particle are to be solved. Then

$$u = \frac{\mu}{h^2} + w\,\cos\theta$$

and

$$\frac{d^2 u}{d\theta^2} = \left(\frac{d^2 w}{d\theta^2} - w\right)\cos\theta - 2\frac{dw}{d\theta}\sin\theta,$$

and therefore

$$\frac{d^2w}{d\theta^2}\,\cos\theta - 2\frac{dw}{d\theta}\,\sin\theta = 0.$$

But then

$$\frac{d}{d\theta}\left(\frac{dw}{d\theta}\cos^2\theta\right) = \frac{d^2w}{d\theta^2}\cos^2\theta - 2\frac{dw}{d\theta}\,\cos\theta\,\sin\theta = 0.$$

It follows that

$$\frac{dw}{d\theta}\,\cos^2\theta = B,$$

where B is a constant. Thus

$$\frac{dw}{d\theta} = \frac{B}{\cos^2\theta} = \frac{d}{d\theta}\left(B\tan\theta\right).$$

It follows that $w = A + B\tan\theta$, where A and B are constants. But then

$$\frac{1}{r} = u = \frac{\mu}{h^2} + A\,\cos\theta + B\sin\theta.$$

Let

$$e = \frac{h^2\sqrt{A^2 + B^2}}{\mu}.$$

Then the point

$$\left(\frac{h^2 A}{e\mu}, \frac{h^2 B}{e\mu}\right)$$

lies on the circle of radius 1 about the origin in the plane, and therefore there exists some real number θ_0 such that

$$\frac{h^2 A}{e\mu} = \cos\theta_0 \quad \text{and} \quad \frac{h^2 B}{e\mu} = \sin\theta_0.$$

Then

$$\frac{h^2}{\mu r} = 1 + e\,\cos\theta\,\cos\theta_0 + e\,\sin\theta\,\sin\theta_0 = 1 + e\cos(\theta - \theta_0),$$

as required. ∎

For simplicity, we can orient the Cartesian coordinate system so that $\theta_0 = 0$. We also let $\ell = h^2/\mu$. Then the equation of the orbit of the particle around the origin becomes

$$\frac{\ell}{r} = 1 + e\,\cos\theta,$$

where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Multiplying both sides of this equation by r, we find that

$$\ell = r + er\cos\theta = r + ex = \sqrt{x^2 + y^2} + ex.$$

It follows that the orbit of the particle has equation

$$x^2 + y^2 = (\ell - ex)^2.$$

Expanding the right hand side of this equation and rearranging, we find that

$$(1 - e^2)x^2 + 2e\ell x + y^2 = \ell^2.$$

The shape of the curve is then determined by the value of the constant e. There are three distinct cases: $e < 1$, $e = 1$, and $e > 1$.

The case when $e = 1$ is the simplest to analyze. In that case the equation of the orbit takes the form

$$y^2 = \ell(\ell - 2x).$$

This curve is a parabola which comes closest to the origin at the point $(\frac{1}{2}\ell, 0)$. Moreover the tangent line at this point of closest approach to the origin is the line $x = \frac{1}{2}\ell$.

Now let us restrict attention to the cases where $e \neq 1$. In those cases

$$\begin{aligned}(1-e^2)\left(x+\frac{e\ell}{1-e^2}\right)^2+y^2 &= \ell^2\left(1+\frac{e^2}{1-e^2}\right)\\ &= \frac{\ell^2}{1-e^2},\end{aligned}$$

and therefore

$$\frac{(x+ae)^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1,$$

where

$$a=\frac{\ell}{1-e^2}.$$

In the case when $0 \leq e < 1$ we can write the equation of the orbit in the form

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b = a\sqrt{1 - e^2}$. This is the equation of an *ellipse* centred on the point $(-ae, 0)$ with *semi-major axis* equal to a and *semi-minor axis* equal to b. The quantity e that determines the shape of the ellipse is known as the *eccentricity* of the ellipse. The *semi-latus rectum* of the ellipse is equal to ℓ, where $\ell = a(1 - e^2)$.
The case where $0 \leq e < 1$ characterizes a closed orbit whose distance from the origin remains bounded. The cases where $e \geq 1$ describe motion in which the particle escapes "to infinity". The following corollary therefore summarizes results we have obtained in analysing the orbit of a particle in the case where the eccentricity e of the orbit satisfies $0 \leq e < 1$.

Corollary 9.7

Let a particle move in a closed orbit in the plane so that its acceleration is always directed towards the origin and is inversely proportional to the square of the distance from the origin, and let a Cartesian coordinate system be oriented so that points where the particle is closest to the origin when it crosses the positive x-axis. Then the orbit of the particle is an ellipse with equation

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$. Moreover if $x = r\cos\theta$ and $y = r\sin\theta$ then r and θ satisfy the equation

$$\frac{(1-e^2)a}{r} = 1 + e\cos\theta.$$