MA1S11—Calculus Portion
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7.6. Integration by Parts

Proposition 7.25 (Integration by Parts)

Let f and g be continuously differentiable real-valued functions on the interval $[a, b]$. Then

$$\int_a^b f(x)\,\frac{dg(x)}{dx}\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,\frac{df(x)}{dx}\,dx.$$

Proof

This result follows from Corollary 7.19 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - g(x)\,\frac{df(x)}{dx}. \qquad \blacksquare$$

Example

We determine the value of

$$\int_0^s x \sin kx \, dx$$

where k is a non-zero real constant. Let

$$f(x) = x \quad \text{and} \quad g(x) = -\frac{1}{k} \cos kx$$

for all real numbers x. Then

$$\frac{dg(x)}{dx} = \sin kx.$$

It follows that

$$\begin{aligned}
\int_0^s x \sin kx \, dx &= \int_0^s f(x) \frac{dg(x)}{dx} \, dx \\
&= [f(x)g(x)]_0^s - \int_0^s \frac{df(x)}{dx} g(x) \, dx \\
&= -\frac{1}{k} [x \cos kx]_0^s + \frac{1}{k} \int_0^s \cos kx \, dx \\
&= -\frac{s}{k} \cos ks + \frac{1}{k^2} [\sin kx]_0^s \\
&= -\frac{s}{k} \cos ks + \frac{1}{k^2} \sin ks.
\end{aligned}$$

Thus

$$\int_0^s x \sin kx \, dx = \frac{1}{k^2} \sin ks - \frac{s}{k} \cos ks.$$

7.7. Integration by Substitution

Proposition 7.26 (Integration by Substitution)

Let $\varphi\colon [a, b] \to \mathbb{R}$ be a continuously-differentiable function on the interval $[a, b]$. Then

$$\int_{\varphi(a)}^{\varphi(b)} f(u)\, du = \int_a^b f(\varphi(x)) \frac{d\varphi(x)}{dx}\, dx.$$

for all continuous real-valued functions f on the range $\varphi([a, b])$ of the function φ.

Proof

Let $c = \varphi(a)$ and $d = \varphi(b)$, and let F and G be the functions on $[a, b]$ defined by

$$F(s) = \int_c^{\varphi(s)} f(u)du, \qquad G(s) = \int_a^s f(\varphi(x))\frac{d\varphi(x)}{dx}dx.$$

Then $F(a) = 0 = G(a)$. Moreover $F(s) = H(\varphi(s))$, where

$$H(w) = \int_c^w f(u)\,du,$$

for all $w \in \varphi([a, b])$. Using the Chain Rule (Proposition 5.5) and the Fundamental Theorem of Calculus (Theorem 7.17), we find that

$$F'(s) = H'(\varphi(s))\varphi'(s) = f(\varphi(s))\varphi'(s) = G'(s)$$

for all $s \in (a, b)$. On applying the Mean Value Theorem (Theorem 5.9) to the function $F - G$ on the interval $[a, b]$, we see that $F(b) - G(b) = F(a) - G(a) = 0$. Thus $H(d) = F(b) = G(b)$, which yields the required identity. ∎

Let x be a real variable taking values in a closed interval $[a, b]$, and let $u = \varphi(x)$ for all $x \in [a, b]$, where $\varphi\colon [a, b] \to \mathbb{R}$ be a continuously-differentiable function on the interval $[a, b]$. The rule for Integration by Substitution (Proposition 7.26) can then be stated as follows:

$$\int_{u(a)}^{u(b)} f(u)\, du = \int_a^b f(u(x))\, \frac{du}{dx}\, dx.$$

for all continuous real-valued functions f whose domain includes $u(x)$ for all real numbers x satisfying $a \leq x \leq b$, where $u(a)$ and $u(b)$ denote the values of u when $x = a$ and $x = b$ respectively.

Example

We determine the value of the integral

$$\int_0^s \frac{x^5}{\sqrt{1-x^2}}\,dx,$$

where s is a real number satisfying $-1 < s < 1$. Let $u = \sqrt{1-x^2}$. Then

$$\frac{du}{dx} = -2x \times \frac{1}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}.$$

Also $u^2 = 1 - x^2$ and therefore $x^2 = 1 - u^2$ and $x^4 = 1 - 2u^2 + u^4$. It follows that

$$\begin{aligned}
\int_0^s \frac{x^5}{\sqrt{1-x^2}}\,dx &= -\int_0^s (1-2u^2+u^4)\,\frac{du}{dx}\,dx \\
&= -\int_{u(0)}^{u(s)} (1-2u^2+u^4)\,du \\
&= -\int_1^{\sqrt{1-s^2}} (1-2u^2+u^4)\,du \\
&= -\left[u-\tfrac{2}{3}u^3+\tfrac{1}{5}u^5\right]_1^{\sqrt{1-s^2}} \\
&= \tfrac{8}{15}-\sqrt{1-s^2}\left(1-\tfrac{2}{3}(1-s^2)+\tfrac{1}{5}(1-s^2)^2\right) \\
&= \tfrac{8}{15}-\sqrt{1-s^2}\left(\tfrac{8}{15}+\tfrac{4}{15}s^2+\tfrac{1}{5}s^4\right).
\end{aligned}$$

Example

We determine the value of the integral

$$\int_0^{\pi} \sin\theta \, \cos^4\theta \, d\theta$$

Let $u = \cos\theta$. Then $\dfrac{du}{d\theta} = -\sin\theta$. It follows that

$$\begin{aligned}
\int_0^{\pi} \sin\theta \, \cos^4\theta \, d\theta &= -\int_0^{\pi} u^4 \frac{du}{d\theta} \, d\theta \\
&= -\int_{\cos(0)}^{\cos(\pi)} u^4 \, du = -\int_1^{-1} u^4 \, du \\
&= \int_{-1}^{1} u^4 \, du = \left[\frac{1}{5}u^5\right]_{-1}^{1} = \frac{1}{5}(1^5 - (-1)^5) \\
&= \frac{2}{5}.
\end{aligned}$$

Example

We determine the value of the integral

$$\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx.$$

Let $u = 1 + x^3$. Then $\dfrac{du}{dx} = 3x^2$. It follows that

$$\begin{aligned}
\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx &= \frac{1}{3}\int_0^2 \frac{1}{\sqrt{u}}\,\frac{du}{dx}\,dx \\
&= \frac{1}{3}\int_1^9 \frac{1}{\sqrt{u}}\,du = \frac{1}{3}\int_1^9 u^{-\frac{1}{2}}\,du \\
&= \frac{1}{3}\left[2u^{\frac{1}{2}}\right]_1^9 = \frac{2}{3}(\sqrt{9}-\sqrt{1}) = \frac{4}{3}.
\end{aligned}$$

Example

We determine the value of the integral

$$\int_0^s x^3 \sin^5(x^4)\cos(x^4)\,dx$$

for all real numbers s. Let $u = \sin(x^4)$. Then

$$\frac{du}{dx} = 4x^3\cos(x^4).$$

Applying the rule for Integration by Substitution, we see that

$$\begin{aligned}\int_0^s x^3 \sin^5(x^4)\cos(x^4)\,dx &= \frac{1}{4}\int_0^s u^5\,\frac{du}{dx}\,dx = \frac{1}{4}\int_{u(0)}^{u(s)} u^5\,du \\ &= \frac{1}{24}\left[u^6\right]_{u(0)}^{u(s)} \\ &= \frac{1}{24}\sin^6(s^4).\end{aligned}$$

Example

We determine the value of the integral

$$\int_1^s \frac{1}{x^2} \sin\left(\frac{2\pi}{x}\right)\,dx$$

for all positive real numbers s. Let $u = \dfrac{2\pi}{x}$. Then

$$\frac{du}{dx} = -\frac{2\pi}{x^2}.$$

It follows that

$$\begin{aligned}
\int_1^s \frac{1}{x^2}\sin\left(\frac{2\pi}{x}\right)\,dx &= -\frac{1}{2\pi}\int_1^s \sin u\,\frac{du}{dx}\,dx \\
&= -\frac{1}{2\pi}\int_{2\pi}^{\frac{2\pi}{s}} \sin u\,du \\
&= \frac{1}{2\pi}\Big[\cos u\Big]_{2\pi}^{\frac{2\pi}{s}} \\
&= \frac{1}{2\pi}\left(\cos\left(\frac{2\pi}{s}\right) - 1\right).
\end{aligned}$$

In the examples we have considered above, we have been given an integral of the form $\int_a^b F(x)\,dx$, and we have evaluated the integral by finding a function u of x and a function $f(u)$ of u for which $F(x) = f(u(x))\dfrac{du}{dx}$. Some calculus texts refer to substitutions of this type as u-substitutions.

In some cases it may be possible to evaluate integrals using the method of Integration by Substitution, but expressing the variable x of integration as a function of some other real variable.

Example

We evaluate

$$\int_0^1 \sqrt{1 - x^2}\, dx.$$

Let $x = \sin\theta$. Then $0 = \sin 0$ and $1 = \sin\frac{1}{2}\pi$. It follows from the rule for Integration by Substitution (Proposition 7.26) that

$$\int_0^1 \sqrt{1 - x^2}\, dx = \int_0^{\frac{1}{2}\pi} \sqrt{1 - \sin^2\theta}\, \frac{d(\sin\theta)}{d\theta}\, d\theta.$$

But $\dfrac{d(\sin\theta)}{d\theta} = \cos\theta$ and $\sqrt{1 - \sin^2\theta} = \cos\theta$ for all real numbers θ satisfying $0 \leq \theta = \frac{1}{2}\pi$. It follows that

$$\int_0^1 \sqrt{1 - x^2}\, dx = \int_0^{\frac{1}{2}\pi} \cos^2\theta\, d\theta.$$

Now $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$. It follows that

$$\int_0^{\frac{1}{2}\pi} \cos^2\theta\,d\theta = \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{1}{2}\pi} = \tfrac{1}{4}(\pi + \sin\pi - \sin 0) = \tfrac{1}{4}\pi.$$

With the benefit of hindsight, this result should not seem too surprising! The curve $y = \sqrt{1-x^2}$ is an arc of a circle representing one quarter of the circle, and the definition of the integral as representing the area between this curve and the x-axis ensures that the integral measures the area of a sector of the unit circle subtending a right angle at the centre of the circle. The area of this sector is then a quarter of the area π of the unit circle.

Example

We determine the value of the integral

$$\int_0^s \frac{1}{(a^2 + x^2)^2}\, dx$$

for all positive real numbers s, where a is a positive real constant. We substitute $x = a\tan\theta$. Let $\beta = \arctan(s/a)$. Then $a\tan\beta = s$. The rule for Integration by Substitution (Proposition 7.26) then ensures that

$$\int_0^s \frac{1}{(a^2 + x^2)^2}\, dx = \int_0^\beta \frac{1}{a^4(1 + \tan^2\theta)^2}\, \frac{d(a\tan\theta)}{d\theta}\, d\theta.$$

Now

$$1 + \tan^2\theta = \sec^2\theta = \frac{1}{\cos^2\theta}.$$

Also

$$\frac{d}{d\theta}(\tan\theta) = \sec^2\theta = \frac{1}{\cos^2\theta}$$

(Corollary 6.14). It follows that

$$\begin{aligned}
\int_0^s \frac{1}{(a^2+x^2)^2}\,dx &= \frac{1}{a^3}\int_0^\beta \cos^4\theta \times \frac{1}{\cos^2\theta}\,d\theta \\
&= \frac{1}{a^3}\int_0^\beta \cos^2\theta\,d\theta = \frac{1}{2a^3}\int_0^\beta (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2a^3}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^\beta = \frac{1}{2a^3}(\beta + \tfrac{1}{2}\sin 2\beta) \\
&= \frac{1}{2a^3}(\beta + \sin\beta\,\cos\beta)
\end{aligned}$$

Now

$$\begin{aligned}\sin\beta\,\cos\beta &= \tan\beta\,\cos^2\beta = \frac{\tan\beta}{1+\tan^2\beta}\\ &= \frac{a^2\tan\beta}{a^2+a^2\tan^2\beta} = \frac{as}{a^2+s^2}.\end{aligned}$$

We conclude therefore that

$$\int_0^s \frac{1}{(a^2+x^2)^2}\,dx = \frac{1}{2a^3}\arctan\left(\frac{s}{a}\right) + \frac{s}{2a^2(a^2+s^2)}.$$

7.8. Indefinite Integrals

Let $f(x)$ be an integrable function of a real variable x. It is commonplace to use the notation $\int f(x)\,dx$ to denote some function $g(x)$ with the property that

$$\frac{d}{dx}(g(x)) = f(x).$$

This function $\int f(x)\,dx$ is said to be an *indefinite integral* of the function f.

It follows from the Fundamental Theorem of Calculus (Theorem 7.17), we find that that if $f(x)$ is an integrable function of x on an interval D, and if a is a real number of D then the function $g(x)$ is an indefinite integral of $f(x)$, where

$$g(x) = \int_a^x f(t)\,dt.$$

We can therefore write $g(x) = \int f(x)\,dx$.

Note that an indefinite integral is only defined up to addition of an arbitrary constant: if $\int f(x)\,dx$ is an indefinite integral of $f(x)$ then so is $\int f(x)\,dx + C$, where C is a real constant known as the *constant of integration*.

7.9. Riemann Sums

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where $a < b$, and let P be a partition of $[a,b]$. Then $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a_0 = x_1 < x_2 < x_2 < \cdots < x_n = b.$$

A *Riemann sum* for the function f on the interval $[a,b]$ is a sum of the form

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),$$

where $x_{i-1} \leq x_i^* \leq x_i$ for $i = 1, 2, \ldots, n$.

The definition of the Darboux lower and upper sums ensures that

$$L(P, f) \leq \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \leq U(P, f)$$

for any Riemann sum $\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$ associated with the partition P. Thus is the partition P is chosen fine enough to ensure that $U(P, f) - L(P, f) < \varepsilon$ then all Riemann sums associated with the partition P and its refinements will differ from one another by at most ε. Moreover if the function f is Riemann-integrable on $[a, b]$, then all Riemann sums associated with the partition P and its refinements will approximate to the value of the integral $\int_a^b f(x)\,dx$ to within an error of at most ε.

Some textbooks use definitions of integration that represent integrals as being, in an appropriate sense, limits of Riemann sums.