MA1S11—Calculus Portion
School of Mathematics, Trinity College
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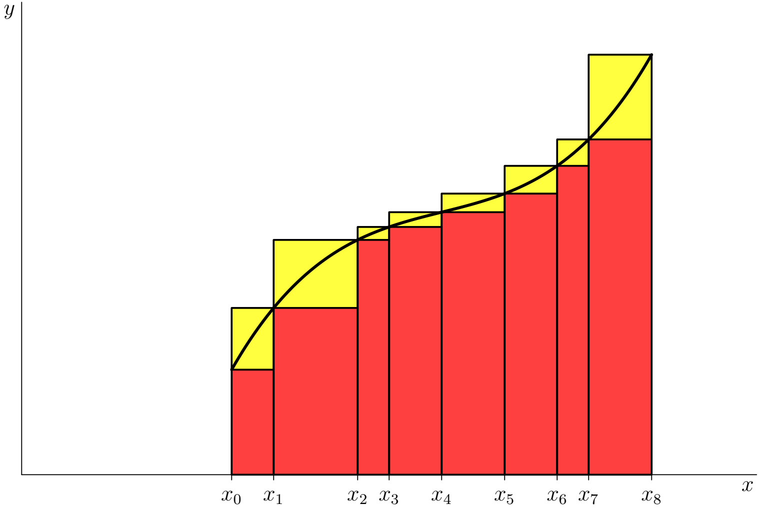
7.3. Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f\colon [a, b] \to \mathbb{R}$ defined on the closed bounded interval $[a, b]$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. Similarly $f\colon [a, b] \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. The function $f\colon [a, b] \to \mathbb{R}$ is said to be *monotonic* on $[a, b]$ if either it is non-decreasing on $[a, b]$ or else it is non-increasing on $[a, b]$.

Proposition 7.14

Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a, b]$ is Riemann-integrable on $[a, b]$.


y
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x_0
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Proof

Let $f\colon [a,b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a,b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, and therefore the function f is bounded on $[a,b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a,b]$ of the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and $x_i - x_{i-1} < \delta$ for $i = 1, 2, \ldots, n$.

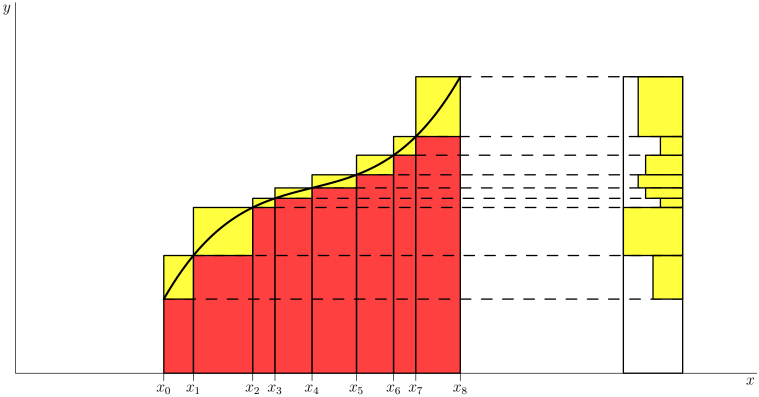
The maximum and minimum values of $f(x)$ on the interval $[x_{i-1}, x_i]$ are attained at x_i and x_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

$$U(P, f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

and

$$L(P, f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Now $f(x_i) - f(x_{i-1}) \geq 0$ for $i = 1, 2, \ldots, n$. It follows that



y
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$$\begin{aligned} & U(P, f) - L(P, f) \\ = & \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ < & \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that, given any positive real number ε, there exists a partition P of the interval $[a, b]$ for which $U(P, f) - L(P, f) < \varepsilon$. It then follows from Corollary 7.6 that the function f is Riemann-integrable on $[a, b]$, as required. ∎

Corollary 7.15

Let a and b be real numbers satisfing $a < b$, and let $f: [a, b] \to \mathbb{R}$ be a real-valued function on the interval $[a, b]$. Suppose that there exist real numbers $x_0, x_1, \ldots, x_n$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

such that the function f restricted to the interval $[x_{i-1}, x_i]$ is monotonic on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then f is Riemann-integrable on $[a, b]$.

Proof

The result follows immediately on applying the results of Proposition 7.11 and Proposition 7.14. ∎

Remark

The result and proof of Proposition 7.14 are to be found in their essentials, though expressed in different language, in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

7.4. Integrability of Continuous functions

The following theorem is stated *without proof*.

Theorem 7.16

Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.

7.5. The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 7.16). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 7.17 (The Fundamental Theorem of Calculus)

Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then

$$\frac{d}{ds}\left(\int_a^s f(x)\,dx\right) = f(s)$$

for all real numbers s satisfying $a < s < b$.

Proof

Let

$$F(t) = \int_a^t f(x)\,dx$$

for all real numbers t satisfying $a \leq t \leq b$. If s is a real number satisfying $a < s < b$, and if h is a real number close enough to zero to ensure that $a \leq h \leq b$ then

$$\begin{aligned} F(s+h) &= \int_a^{s+h} f(x)\,dx = \int_a^s f(x)\,dx + \int_s^{s+h} f(x)\,dx \\ &= F(s) + \int_s^{s+h} f(x)\,dx \end{aligned}$$

(see Corollary 7.12). Also $\displaystyle\int_s^{s+h} c\,dx = hc$ for all real constants c.
It follows that

$$F(s+h) - F(s) - hf(s) = \int_s^{s+h} \bigl(f(x) - f(s)\bigr)\,dx$$

for all real numbers s satisfying $a < s < b$ and for all real numbers h close enough to zero to ensure that $a \leq s + h \leq b$.

Let s be a real number satisfying $a < s < b$, and let some strictly positive real number ε be given. Let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at s, where $a < s < b$. It follows that there exists some strictly positive real number δ such that
$a \leq x \leq b$ and

$$f(s) - \varepsilon_0 \leq f(x) \leq f(s) + \varepsilon_0$$

for all real numbers x satisfying $s - \delta < x < s + \delta$.

Now

$$-\varepsilon_0 \leq f(x) - f(s) \leq \varepsilon_0$$

for all real numbers x that lie between s and $s + h$. It follows that

$$-\varepsilon_0 |h| \leq \int_s^{s+h} \bigl(f(x) - f(s)\bigr)\, dx \leq \varepsilon_0 |h|$$

for all real numbers h satisfying $0 < |h| < \delta$, Also

$$\frac{F(s+h) - F(s)}{h} - f(s) = \frac{1}{h} \int_s^{s+h} \bigl(f(x) - f(s)\bigr)\, dx$$

for all real numbers h satisfying $0 < |h| < \delta$. It follows that

$$-\varepsilon < -\varepsilon_0 \leq \frac{F(s+h) - F(s)}{h} - f(s) \leq \varepsilon_0 < \varepsilon$$

for all real numbers h satisfying $0 < |h| < \delta$.

We conclude from this that

$$F'(s) = \left.\frac{dF(x)}{dx}\right|_{x=s} = \lim_{h\to 0} \frac{F(s+h)-F(s)}{h} = f(s),$$

as required. ∎

Corollary 7.18

Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then

$$\frac{d}{ds}\left(\int_s^b f(x)\,dx\right) = -f(s)$$

for all real numbers s satisfying $a < s < b$.

Proof

The integral satisfies

$$\int_a^b f(x)\,dx = \int_a^s f(x)\,dx + \int_s^b f(x)\,dx.$$

(see Proposition 7.11).

Differentiating this identity, and applying the Fundamental Theorem of Calculus (Theorem 7.17), we find that

$$\begin{aligned} 0 &= \frac{d}{ds}\left(\int_a^b f(x)\,dx\right) \\ &= \frac{d}{ds}\left(\int_a^s f(x)\,dx\right) + \frac{d}{ds}\left(\int_s^b f(x)\,dx\right) \\ &= f(s) + \frac{d}{ds}\left(\int_s^b f(x)\,dx\right). \end{aligned}$$

The result follows. ∎

Let $f: [a, b] \to \mathbb{R}$ be a continuous function on a closed interval $[a, b]$. We say that f is *continuously differentiable* on $[a, b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$\begin{aligned} f'(a) &= \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \\ f'(b) &= \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \end{aligned}$$

exist at the endpoints of $[a, b]$, and the function f' is continuous on $[a, b]$.

If $f\colon [a,b] \to \mathbb{R}$ is continuous, and if $F(s) = \int_a^x f(x)\,dx$ for all $s \in [a,b]$ then the one-sided derivatives of F at the endpoints of $[a,b]$ exist, and

$$\lim_{h\to 0^+} \frac{F(a+h)-F(a)}{h} = f(a), \qquad \lim_{h\to 0^-} \frac{F(b+h)-F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Corollary 7.19

Let f be a continuously-differentiable real-valued function on a closed interval with endpoints a and b. Then

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a).$$

Proof

The result in the case when $b < a$ follows from that in the case when $a < b$ by interchanging the limits a and b of integration, since both sides of the identity change sign when a and b are interchanged. It therefore suffices to prove the result in the case when $a < b$.

Define $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\,dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\,dt\right) = 0$$

for all x satisfing $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 5.9) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b-a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

When evaluating definite integrals, it is customary to denote the difference in the values of a function between the endpoints of an interval by $[f(x)]_a^b$, where

$$[f(x)]_a^b = f(b) - f(a).$$

The result of Corollary 7.19 is therefore represented by the following identity: valid for all continuously-differentiable functions f on a closed interval with endpoints a and b:

$$\int_a^b \frac{df(x)}{dx}\,dx = [f(x)]_a^b = f(b) - f(a).$$

Corollary 7.20

Let q be a rational number, where $q \neq -1$. Then

$$\int_a^b x^q \, dx = \frac{1}{q+1}(b^{q+1} - a^{q+1})$$

for all positive real numbers a and b. Moreover this identity is valid for all real numbers a and b in the special case where q is a non-negative integer.

Proof

Applying Corollary 7.19, we find that

$$\begin{aligned} \int_a^b x^q \, dx &= \frac{1}{q+1} \int_a^b \frac{d}{dx}\left(x^{q+1}\right) \, dx = \frac{1}{q+1}\left[x^{q+1}\right]_a^b \\ &= \frac{1}{q+1}(b^{q+1} - a^{q+1}), \end{aligned}$$

as required. ∎

Corollary 7.21

Let $p(x)$ be a polynomial function of a real variable x, and let

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n,$$

where $c_0, c_1, c_2, c_3, \ldots, c_n$ are real constants. Then

$$\int_a^b p(x)\,dx = \Big[P(x) \Big]_a^b,$$

for all real numbers a and b, where

$$P(x) = c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \cdots + \frac{c_n}{n+1} x^{n+1}.$$

Proof
Applying Corollary 7.19, we find that

$$\int_a^b p(x)\,dx = \int_a^b \frac{dP(x)}{dx}\,dx = [P(x)]_a^b = P(b) - P(a),$$

as required. ∎

Example

We determine the value of

$$\int_1^3 (6x^2 - 4x + 3)\,dx$$

Applying Corollary 7.21 we find that

$$\begin{aligned}
&\int_1^3 (6x^2 - 4x + 3)\,dx \\
&\quad = \left[2x^3 - 2x^2 + 3x\right]_1^3 \\
&\quad = (2 \times 3^3 - 2 \times 3^2 + 3 \times 3) - (2 \times 1^3 - 2 \times 1^2 + 3 \times 1) \\
&\quad = 42
\end{aligned}$$

Corollary 7.22

Let k be a real number. Then

$$\int_0^s \sin kx\, dx = \frac{1}{k}(1 - \cos ks),$$

and

$$\int_0^s \cos kx\, dx = \frac{1}{k}\sin ks.$$

Proof

Applying Corollary 7.19, we find that

$$\begin{aligned}\int_0^s \sin kx\,dx &= -\frac{1}{k}\int_0^s \frac{d}{dx}(\cos kx)\,dx = -\frac{1}{k}\left[\cos kx\right]_0^s \\ &= \frac{1}{k}(1-\cos ks)\end{aligned}$$

and

$$\begin{aligned}\int_0^s \cos kx\,dx &= \frac{1}{k}\int_0^s \frac{d}{dx}(\sin kx)\,dx = \frac{1}{k}\left[\sin kx\right]_0^s \\ &= \frac{1}{k}\sin ks,\end{aligned}$$

as required. ∎

Corollary 7.23

Let a be a real number. Then

$$\int_0^s \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{s}{a}\right).$$

Proof

The derivative of the inverse tangent function satisfies

$$\frac{d}{du}(\arctan u) = \frac{1}{1+u^2}$$

(see Proposition 6.16).

It follows from the Chain Rule (Proposition 5.5) that

$$\frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{1}{a} \times \frac{1}{1 + \dfrac{x^2}{a^2}} = \frac{a}{a^2 + x^2}.$$

Applying Corollary 7.19, we now find that

$$\begin{aligned} \int_0^s \frac{1}{a^2 + x^2}\,dx &= \frac{1}{a}\int_0^s \frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right)\,dx \\ &= \frac{1}{a}\left[\arctan\left(\frac{x}{a}\right)\right]_0^s \\ &= \frac{1}{a}\arctan\left(\frac{s}{a}\right), \end{aligned}$$

as required. $\blacksquare$

Corollary 7.24

Let a be a real number. Then

$$\int_0^s \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{s}{a}\right).$$

Proof

The derivative of the inverse sine function satisfies

$$\frac{d}{du}(\arcsin u) = \frac{1}{\sqrt{1 - u^2}}$$

(see Proposition 6.18).

It follows from the Chain Rule (Proposition 5.5) that

$$\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) = \frac{1}{a} \times \frac{1}{\sqrt{1 - \dfrac{x^2}{a^2}}} = \frac{1}{\sqrt{a^2 - x^2}}.$$

Applying Corollary 7.19, we now find that

$$\begin{aligned} \int_0^s \frac{1}{\sqrt{a^2 - x^2}}\, dx &= \int_0^s \frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) dx \\ &= \left[\arcsin\left(\frac{x}{a}\right)\right]_0^s \\ &= \arcsin\left(\frac{s}{a}\right), \end{aligned}$$

as required. ∎