MA1S11—Calculus Portion
School of Mathematics, Trinity College
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We recall the basic definitions associated with the definition of the Riemann integral (or Riemann-Darboux) integral of a bounded real-valued function $f\colon [a, b] \to \mathbb{R}$ on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a < b$. The function f is required to be bounded, and therefore there exist real numbers m and M with the property that $m \leq f(x) \leq M$ for all real numbers x satisfying $a \leq x \leq b$.

A *partition* P of the interval $[a, b]$, may be specified in the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $x_0, x_1, \ldots, x_n$ are real numbers satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

The quantities m_i and M_i are defined for $i = 1, 2, \ldots, n$ so that

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

and

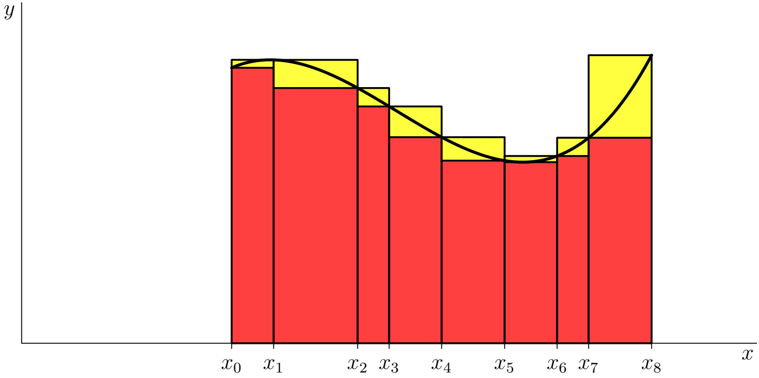
$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

Then the interval $[m_i, M_i]$ can be characterized as the smallest closed interval in $\mathbb{R}$ that contains the set

$$\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

The *Darboux lower sum* $L(P, f)$ and *Darboux upper sum* $U(P, f)$ determined by the function f and the partition P of the interval $[a, b]$ are then defined by the identities

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$



y
x
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8

7. Integration (continued)

The *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ of the function f on the interval $[a, b]$ is defined to be the least upper bound of the Darboux lower sums $L(P, f)$ as P ranges over all partitions of the interval $[a, b]$.

Similarly the *upper Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ of the function f on the interval $[a, b]$ is defined to be the greatest lower bound of the Darboux upper sums $U(P, f)$ as P ranges over all partitions of the interval $[a, b]$.

The lower and upper Riemann integrals of the function f on the interval $[a, b]$ are therefore characterized by the properties presented in the following lemmas.

Lemma 7.1

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a < b$. Then the lower Riemann integral is the unique real number characterized by the following two properties:—

(i)

$$L(P,f) \leq \mathcal{L}\int_a^b f(x)\,dx$$

for all partitions P of the interval $[a,b]$.

(ii) *given any positive real number ε, there exists a partition P of the interval $[a,b]$ for which*

$$L(P,f) > \mathcal{L}\int_a^b f(x)\,dx - \varepsilon.$$

Lemma 7.2

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a < b$. Then the upper Riemann integral is the unique real number characterized by the following two properties:—

(i)

$$U(P,f) \geq \mathcal{U}\int_a^b f(x)\,dx$$

for all partitions P of the interval $[a,b]$.

(ii) *given any positive real number ε, there exists a partition P of the interval $[a,b]$ for which*

$$U(P,f) < \mathcal{U}\int_a^b f(x)\,dx + \varepsilon.$$

A bounded function f on the interval $[a, b]$ is then *Riemann-integrable* if and only if

$$\mathcal{L}\int_a^b f(x)\,dx = \mathcal{U}\int_a^b f(x)\,dx.$$

The *integral* $\int_a^b f(x)\,dx$ of a Riemann-integrable function f on the interval $[a, b]$ is then the common value of the upper and lower Riemann integrals.

In order to develop further the theory of integration, we introduce the notion of a *refinement* of a partition, and prove that if we replace a partition P by a refinement R of that partition, then the Darboux upper and lower sums satisfy the inequalities

$$L(R, f) \geq L(P, f) \quad \text{and} \quad U(R, f) \leq U(P, f).$$

for all bounded functions f on $[a, b]$. This result is an essential tool in developing the theory of the Riemann integral.

Definition

Let P and R be partitions of $[a, b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 7.3

Let R be a refinement of some partition P of $[a, b]$. Then

$$L(R, f) \geq L(P, f) \quad \textit{and} \quad U(R, f) \leq U(P, f)$$

for any bounded function $f\colon [a, b] \to \mathbb{R}$.

Proof

Let $P = \{x_0, x_1, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$L(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

and

$$U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

and

$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

Suppose that we add an extra division point to P to obtain a partition Q. We suppose that the extra division point z is added between x_{k-1} and x_k, where k is some integer between 1 and n, so that $x_{k-1} < z < x_k$. Let

$$\begin{aligned} m'_k &= \inf\{f(x) \mid x_{k-1} \leq x \leq z\}, \\ M'_k &= \sup\{f(x) \mid x_{k-1} \leq x \leq z\}, \\ m''_k &= \inf\{f(x) \mid z \leq x \leq x_k\}, \\ M''_k &= \sup\{f(x) \mid z \leq x \leq x_k\}. \end{aligned}$$

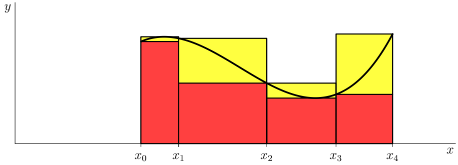
Then $m_k \leq m'_k$, $m_k \leq m''_k$, $M_k \geq M'_k$ and $M_k \geq M''_k$.

It follows that

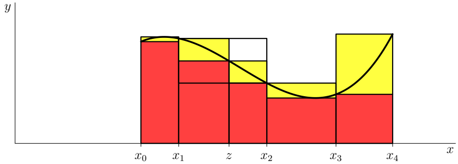
$$\begin{aligned} m_k(x_k - x_{k-1}) &= m_k(z - x_{k-1}) + m_k(x_k - z) \\ &\leq m'_k(z - x_{k-1}) + m''_k(x_k - z) \end{aligned}$$

and

$$\begin{aligned} M_k(x_k - x_{k-1}) &= M_k(z - x_{k-1}) + M_k(x_k - z) \\ &\geq M'_k(z - x_{k-1}) + M''_k(x_k - z) \end{aligned}$$



Darboux sums before refinement



Darboux sums with new division point z between x_1 and x_2

But the lower sum $L(P, f)$ is the sum of the quantities $m_i(x_i - x_{i-1})$ as i ranges from 1 to n, and the lower sum $L(Q, f)$ is the analogous sum for the partition Q, obtained on replacing the summand $m_k(x_k - x_{k_1})$ by the quantity

$$m'_k(z - x_{k-1}) + m''_k(x_k - z),$$

which is no smaller than $m_k(x_k - x_{k_1})$. It follows that $L(P, f) \leq L(Q, f)$.

Similarly $U(P, f)$ is the sum of the quantities $M_i(x_i - x_{i-1})$ as i ranges from 1 to n, and the upper sum $U(Q, f)$ is the analogous sum for the partition Q, obtained on replacing the summand $M_k(x_k - x_{k-1})$ by the quantity

$$M'_k(z - x_{k-1}) + M''_k(x_k - z),$$

which is no larger than $M_k(x_k - x_{k_1})$. It follows that $U(P, f) \geq U(Q, f)$.

If the partition R of the interval $[a, b]$ is a refinement of the partition P, then one can obtain R from P by successively adding extra division points, one at a time. We have shown that the lower sums do not decrease, and the upper sums do not increase, each time a new division point is added. It follows that

$$L(R, f) \geq L(P, f) \quad \text{and} \quad U(R, f) \leq U(P, f),$$

as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. Indeed we can take R to be the partition of $[a, b]$ obtained in taking as division points all the division points belonging to the partitions P and Q. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 7.4

Let f be a bounded real-valued function on the interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Then

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof

Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 7.3 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L} \int_a^b f(x)\, dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L} \int_a^b f(x)\, dx \leq \mathcal{U} \int_a^b f(x)\, dx$, as required. ∎

Proposition 7.5

Let f be a bounded real-valued function on the interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Then the function f is Riemann-integrable on f, with Riemann integral $\int_a^b f(x)\,dx$ if and only if the following two properties are satisfied:

(i)

$$L(P, f) \le \int_a^b f(x)\,dx \le U(P, f)$$

for all partitions P of the interval $[a, b]$;

(ii) *given any positive real number ε, there exists a partition P of the interval $[a, b]$ for which*

$$\int_a^b f(x)\,dx - \varepsilon < L(P, f) \le U(P, f) < \int_a^b f(x) + \varepsilon.$$

Proof

Let A be a real number. Suppose that $L(P, f) \leq A \leq U(P, f)$ for all partitions P of $[a, b]$, and that, given any positive real number ε, there exists a partition P of $[a, b]$ for which $A - \varepsilon < L(P, f) \leq U(P, f) < A + \varepsilon$. It then follows from Lemma 7.1 and Lemma 7.2 that $A = \mathcal{L} \int_a^b f(x)\, dx$ and $A = \mathcal{U} \int_a^b f(x)\, dx$. Therefore the function f is Riemann-integrable on $[a, b]$, and $\int_a^b f(x)\, dx = A$.

Conversely, suppose that the function f is Riemann-integrable on $[a,b]$, with Riemann integral equal to the real number A. Then $A = \mathcal{L}\int_a^b f(x)\,dx = \mathcal{U}\int_a^b f(x)\,dx$, and therefore $L(P,f) \leq A \leq U(P,f)$ for all partitions P of $[a,b]$. Moreover it follows from Lemma 7.1 and Lemma 7.2 that there exist partitions P_1 and P_2 of $[a,b]$ for which $L(P_1,f) > A - \varepsilon$ and $U(P_2,f) < A + \varepsilon$. Let P be a common refinement of the partitions P_1 and P_2. It follows from Lemma 7.3 that

$$A - \varepsilon < L(P_1,f) \leq L(P,f) \leq U(P,f) < U(P_2,f) < A + \varepsilon.$$

The result follows. ∎

Corollary 7.6

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfing $a \leq b$. Then the function f is Riemann-integrable on $[a,b]$ if and only if, given any positive real number ε, there exists a partition P of $[a,b]$ with the property that

$$U(P,f) - L(P,f) < \varepsilon.$$

Proof

Suppose that the bounded function f is Riemann-integrable on $[a,b]$. Let $A = \int_a^b f(x)\,dx$. It follows from Proposition 7.5 that, given any positive real number ε, there exists a partition P of $[a,b]$ for which

$$A - \tfrac{1}{2}\varepsilon < L(P,f) \leq U(P,f) < A + \tfrac{1}{2}\varepsilon.$$

Then $U(P,f) - L(P,f) < \varepsilon$.

Conversely suppose that f is a bounded function on $[a, b]$ for which there exists a partition P with $U(P, f) - L(P, f) < \varepsilon$. Then

$$L(P, f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P, f)$$

(see Lemma 7.4). Therefore

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all positive real numbers ε. But the difference of the upper and lower Riemann integrals is independent of ε. It follows that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx = 0,$$

and thus the function f is Riemann-integrable on $[a, b]$, as required. ∎

Corollary 7.7

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed bounded interval $[a,b]$, where $a < b$, and let u and v be real numbers belonging to $[a,b]$. Then the function f is Riemann-integrable on the interval with endpoints u and v, and

$$\int_u^v f(x)\,dx = -\int_v^u f(x)\,dx.$$

Proof

First suppose that $a \le u < v \le b$. Let some positive real number ε be given. Then there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Let Q be the partition of $[u,v]$ consisting of the endpoints u and v of the closed interval $[u,v]$ together with those division points of P that lie in the interior of this interval.

An examination of the relevant definitions shows that

$$U(Q, f) - L(Q, f) \leq U(P, f) - L(P, f) < \varepsilon.$$

It follows that if $a \leq u < v \leq b$ then the function f is Riemann-integrable on $[u, v]$. The definition of the relevant integrals then ensures that

$$\int_v^u f(x)\,dx = -\int_u^v f(x)\,dx.$$

(see subsection 7.2).
If $a \leq v < u \leq b$ then the required result follows from the case already proved on interchanging u and v. If $a \leq u = v \leq b$ then the integrals $\int_u^v f(x)\,dx$ and $\int_v^u f(x)\,dx$ are equal to zero, and therefore the result follows in this case also. This completes the proof. ∎

Lemma 7.8

Let $f \colon [a, b] \to \mathbb{R}$ and $g \colon [a, b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed interval $[a, b]$, where $a < b$. Suppose that $f(x) \leq g(x)$ for all real numbers x satisfying $a \leq x \leq b$. Then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proof

The Darboux lower and uppse sums of the functions f and g satisfy $L(P, f) \leq L(P, g)$ and $U(P, f) \leq U(P, g)$. It follows that

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{L}\int_a^b g(x)\,dx \quad \text{and} \quad \mathcal{U}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b g(x)\,dx.$$

The result follows. ∎

Proposition 7.9

Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f+g$ and $f-g$ are Riemann-integrable on $[a,b]$, and moreover

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof
Let $\int_a^b f(x)\,dx = A'$ and $\int_a^b g(x)\,dx = A''$, and let $A = A' + A''$. Let some positive number ε be given. It follows from Proposition 7.5 that there exist partitions P' and P'' of $[a, b]$ that satisfy

$$A' - \tfrac{1}{2}\varepsilon < L(P', f) < U(P', f) < A' + \tfrac{1}{2}\varepsilon$$

and

$$A'' - \tfrac{1}{2}\varepsilon < L(P'', g) < U(P'', g) < A'' + \tfrac{1}{2}\varepsilon$$

Let P be a common refinement of the partitions P' and P''. Then $L(P', f) \le L(P, f)$, $L(P'', g) \le L(P, g)$, $U(P', f) \ge U(P, f)$ and $U(P'', g) \ge L(P, g)$, and therefore

$$A - \varepsilon < L(P, f) + L(P, g) \le U(P, f) + U(P, g) < A + \varepsilon.$$

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let

$$\begin{aligned}
M_i' &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i'' &= \sup\{g(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i &= \sup\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}, \\
m_i' &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\
m_i'' &= \inf\{g(x) : x_{i-1} \leq x \leq x_i\}, \\
m_i &= \inf\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.
\end{aligned}$$

Let i be an integer between 1 and n. Then $m_i' \leq f(x) \leq M_i'$ and $m_i'' \leq g(x) \leq M_i''$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$, and therefore

$$m_i' + m_i'' \leq f(x) + g(x) \leq M_i' + M_i''$$

for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$. It follows that

$$m_i' + m'' \leq m_i \leq M_i \leq M_i' + M''$$

for $i = 1, 2, \ldots, n$. Multiplying by $x_i - x_{i-1}$ and summing over i, we find that

$$\begin{aligned}
\sum_{i=1}^{n} m_i'(x_i - x_{i-1}) + \sum_{i=1}^{n} m_i''(x_i - x_{i-1}) & \\
&\leq \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} M_i'(x_i - x_{i-1}) + \sum_{i=1}^{n} M_i''(x_i - x_{i-1}).
\end{aligned}$$

Thus

$$\begin{aligned} L(P, f) + L(P, g) &\leq L(P, f+g) \\ &\leq U(P, f+g) \leq U(P, f) + U(P, g). \end{aligned}$$

It then follows from inequalities obtained earlier in the proof that

$$A - \varepsilon < L(P, f+g) \leq L(P, f+g) < A + \varepsilon.$$

The result therefore follows on applying Proposition 7.5 to the function $f + g$ on $[a, b]$. $\blacksquare$

Lemma 7.10

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed bounded interval $[a,b]$, where $a < b$, and let c be a real number. Then cf is Riemann-integrable on $[a,b]$, and

$$\int_a^b (cf(x))\,dx = c\int_a^b f(x)\,dx.$$

Proof

Let $A = \int_a^b f(x)\,dx$. The result is immediate if $c = 0$. Suppose that $c > 0$. Then $L(P, cf) = cL(P, f)$ and $U(P, cf) = cU(P, f)$ for all partitions P of $[a, b]$. It follows that $L(P, cf) \leq cA \leq U(P, cf)$ for all partitions P of $[a, b]$. Also, given any positive real number ε, there exists a partition P of $[a, b]$ for which

$$A - \varepsilon/c < L(P, f) \leq U(P, f) < A + \varepsilon/c$$

(see Proposition 7.5). But then

$$cA - \varepsilon < L(P, cf) \leq U(P, cf) < cA + \varepsilon.$$

The result therefore follows in the case when $c > 0$.

The result is also true in the case where $c = -1$, because $L(P, -f) = -U(P, f)$ and $U(P, -f) = -L(P, f)$ for all partitions P of the interval $[a, b]$. Combining these results, we see that the result is true for all real numbers c, as required. ∎

Proposition 7.11

Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof

Let some positive real number ε be given. There exist partitions P_1 and P_2 of $[a, b]$ and $[b, c]$ respectively for which

$$\int_a^b f(x)\,dx - \tfrac{1}{4}\varepsilon < L(P_1, f) \leq U(P_1, f) < \int_a^b f(x)\,dx + \tfrac{1}{4}\varepsilon$$

$$\int_b^c f(x)\,dx - \tfrac{1}{4}\varepsilon < L(P_2, f) \leq U(P_2, f) < \int_b^c f(x)\,dx + \tfrac{1}{4}\varepsilon$$

(see Proposition 7.5).

The partitions P_1 and P_2 combine to give a partition P of $[a, c]$, where $P = P_1 \cup P_2$. Moreover

$$L(P, f) = L(P_1, f) + L(P_2, f) \quad \text{and} \quad U(P, f) = U(P_1, f) + U(P_2, f).$$

It follows that

$$\begin{aligned} \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \tfrac{1}{2}\varepsilon & \\ &< L(P, f) \leq U(P, f) \\ &< \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \tfrac{1}{2}\varepsilon, \end{aligned}$$

and therefore $U(P, f) - L(P, f) < \varepsilon$. It now follows from Corollary 7.6 that the function f is Riemann-integrable in $[a, b]$, and then follows from Proposition 7.5 that

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx,$$

as required. ∎

Corollary 7.12

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a,b]$, where $a < b$. Then

$$\int_u^w f(x)\,dx = \int_u^v f(x)\,dx + \int_v^w f(x)\,dx$$

for all real numbers u, v and w belonging to $[a,b]$.

Proof

In the case where $u = w$, the result follows from the identity

$$\int_v^u f(x)\,dx = -\int_u^v f(x)\,dx$$

(see Lemma 7.7). In the case where $u = v$ and when $v = w$ the result follows from the definition of the integral, which requires that $\int_u^u f(x)\,dx = 0$ and $\int_w^w f(x)\,dx = 0$.

In the case when $u < v < w$, the result follows directly from Proposition 7.11. In the case when $u < w < v$, it follows from Proposition 7.11 that

$$\int_u^v f(x)\,dx = \int_u^w f(x)\,dx + \int_w^v f(x)\,dx$$

It then follows that

$$\begin{aligned}\int_u^w f(x)\,dx &= \int_u^v f(x)\,dx - \int_w^v f(x)\,dx \\ &= \int_u^v f(x)\,dx + \int_v^w f(x)\,dx.\end{aligned}$$

It then follows that if either $v < w < u$ or $v < u < w$ then

$$\int_v^u f(x)\,dx = \int_v^w f(x)\,dx + \int_w^u f(x)\,dx,$$

and therefore

$$\begin{aligned}\int_u^w f(x)\,dx &= -\int_w^u f(x)\,dx \\ &= -\int_v^u f(x)\,dx + \int_v^w f(x)\,dx \\ &= \int_u^v f(x)\,dx + \int_v^w f(x)\,dx.\end{aligned}$$

Finally if $w < u < v$ or $w < v < u$ then

$$\int_w^v f(x)\,dx = \int_w^u f(x)\,dx + \int_u^v f(x)\,dx,$$

and therefore

$$\begin{aligned}\int_u^w f(x)\,dx &= -\int_w^u f(x)\,dx \\ &= \int_u^v f(x)\,dx - \int_w^v f(x)\,dx \\ &= \int_u^v f(x)\,dx + \int_v^w f(x)\,dx.\end{aligned}$$

This completes the proof. ∎

Proposition 7.13

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Riemann-integrable real-valued function on a closed bounded interval $[a,b]$, where $a < b$, and let k be a positive real number. Then

$$\int_a^b f(x)\,dx = k \int_{a/k}^{b/k} f(ku)\,du.$$

Proof

Let $g\colon [a/k, b/k] \to \mathbb{R}$ be defined so that $g(u) = f(ku)$ for all real numbers u satisfying $a/k \leq u \leq b/k$. Each partition P of $[a, b]$ determines a corresponding partition Q of $[a/k, b/k]$ so that if $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

then $Q = \{u_0, u_1, \ldots, u_n\}$, where $u_i = x_i/k$ for $i = 1, 2, \ldots, n$. Then $kL(Q, g) = L(P, f)$ and $kU(Q, g) = U(P, f)$. This ensures that

$$k \int_{a/k}^{b/k} f(ku)\, du = k \int_{a/k}^{b/k} g(u)\, du = \int_a^b f(x)\, dx,$$

as required. ∎