MA1S11—Calculus Portion
School of Mathematics, Trinity College
Michaelmas Term 2016
Lecture 24 (November 29, 2016)

David R. Wilkins

7. Integration

7.1. Darboux Sums of a Bounded Function

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function on a closed interval $[a,b]$ that is bounded above and below on the interval $[a,b]$, where a and b are real numbers satisfying $a < b$. Then there exist real numbers m and M such that $m \leq f(x) \leq M$ for all real numbers x satisfying $a \leq x \leq b$. We seek to define a quantity $\int_a^b f(x)\,dx$, the *definite integral* of the function f on the interval $[a,b]$, where the value of this quantity represents the area "below" the graph of the function where the function is positive, minus the area "above" the graph of the function where the function is negative.

We now introduce the definition of a *partition* of the interval $[a, b]$.

Definition

A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

A partition P of the closed interval $[a, b]$ provides a decomposition of that interval as a union of the subintervals $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Successive subintervals of the partition intersect only at their endpoints.

Let P be a partition of the interval $[a, b]$. Then $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where $x_0, x_1, \ldots, x_n$ are real numbers satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The values of the bounded function $f\colon [a, b] \to \mathbb{R}$ satisfy $m \leq f(x) \leq M$ for all real numbers x satisfying $a \leq x \leq b$. It follows that, for each integer i between i and n, the set

$$\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

is a set of real numbers that is bounded below by m and bounded above by M. The Least Upper Bound Principle then ensures that the set $\{f(x) \mid x_{i-1} \leq x \leq x_i\}$ has a well-defined greatest lower bound and a well-defined least upper bound (see the discussion of least upper bounds and greatest lower bounds in Subsections 1.15 to 1.19).

For each integer i between 1 and n, let us denote by m_i the greatest lower bound on the values of the function f on the interval $[x_{i-1}, x_i]$, and let us denote by M_i the least upper bound on the values of the function f on the interval $[x_{i-1}, x_i]$, so that

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

and

$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

Then the interval $[m_i, M_i]$ can be characterized as the smallest closed interval in $\mathbb{R}$ that contains the set

$$\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

We now consider what the values of the greatest lower bound and least upper bound on the values of the function are determined in particular cases where the function has some special behaviour.

First suppose that the function f is non-decreasing on the interval $[x_{i-1}, x_i]$. Then $m_i = f(x_{i-1})$ and $M_i = f(x_i)$, because in this case the values of the function f satisfy $f(x_{i-1}) \leq f(x) \leq f(x_i)$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$.

Next suppose that the function f is non-increasing on the interval $[x_{i-1}, x_i]$. Then $m_i = f(x_i)$ and $M_i = f(x_{i-1})$, because in this case the values of the function f satisfy $f(x_{i-1}) \geq f(x) \geq f(x_i)$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$.

Next suppose that that the function f is continuous on the interval $[x_{i-1}, x_i]$. The Extreme Value Theorem (Theorem 4.29) then ensures the existence of real numbers u_i and v_i, where $x_{i-1} \leq u_i \leq x_i$ and $x_{i-1} \leq v_i \leq x_i$ with the property that

$$f(u_i) \leq f(x) \leq f(v_i)$$

for all real numbers x satisfying $x_{i-1} \leq u_i \leq x_i$. Then $m_i = f(u_i)$ and $M_i = f(v_i)$.

Finally consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined such that $f(x) = x - \lfloor x \rfloor$ for all real numbers x, where $\lfloor x \rfloor$ is the greatest integer satisfying the inequality $\lfloor x \rfloor \leq x$. Then $0 \leq f(x) < 1$ for all real numbers x. If the interval $[x_{i-1}, x_i]$ includes an integer in its interior then

$$\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} = 1,$$

and thus $M_i = 1$, even though there is no real number x for which $f(x) = 1$.

We now summarize the essentials of the discussion so far. The function $f \colon [a, b] \to \mathbb{R}$ is a bounded function on the closed interval $[a, b]$, where a and b are real numbers satisfying $a < b$. There then exist real numbers m and M such that $m \leq f(x) \leq M$ for all real numbers x satisfying $a \leq x \leq b$. We are given also a partition P of the interval $[a, b]$. This partition P is representable as a finite set of real numbers in the interval $[a, b]$ that includes the endpoints of the interval. Thus

$$P = \{x_0, x_1, \ldots, x_n\}$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The quantities m_i and M_i are then defined so that

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

and

$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

for $i = 1, 2, \ldots, n$. Then $m_i \leq f(x) \leq M_i$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$. Moreover $[m_i, M_i]$ is the smallest closed interval that contains all the values of the function f on the interval $[x_{i-1}, x_i]$.

Definition

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function defined on a closed bounded interval $[a,b]$, where $a < b$, and let the partition P be a partition of $[a,b]$ given by $P = \{x_0, x_1, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Then the *lower sum* (or *lower Darboux sum*) $L(P,f)$ and the *upper sum* (or *upper Darboux sum*) $U(P,f)$ of f for the partition P of $[a,b]$ are defined so that

$$L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \qquad U(P,f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}$.

Clearly $L(P,f) \leq U(P,f)$. Moreover $\sum_{i=1}^{n}(x_i - x_{i-1}) = b - a$, and therefore

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a,b]$.

Remark

Let us consider how the lower and upper sum of a bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ are related to the notion of the area "under the graph of the function f" on the interval a, in the case where the function f is non-negative on the interval $[a,b]$. Thus suppose that $f(x) \geq 0$ for all $x \in [a,b]$, and let X denote the region of the plane bounded by the graph of the function f from $x = a$ to $x = b$ and the lines $x = a$, $x = b$ and $y = 0$. Then

$$X = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\},$$

where $\mathbb{R}^2$ is the set of all ordered pairs of real numbers. (The elements of $\mathbb{R}^2$ are then regarded as Cartesian coordinates of points of the plane.)

For each integer i let

$$\begin{aligned} X_i &= \{(x,y) \in X \mid x_{i-1} \leq x \leq x_i\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x_{i-1} \leq x \leq x_i \text{ and } 0 \leq y \leq f(x)\}. \end{aligned}$$

If the regions X and X_i have well-defined areas for $i = 1, 2, \ldots, n$ satisfying the properties that areas of planar regions are expected to satisfy, then

$$\mathrm{area}(X) = \sum_{i=1}^{n} \mathrm{area}(X_i),$$

because, where subregions X_i for different values of i intersect one another, they intersect only along their bounding edges.

Let i be an integer between 1 and n. Then $0 \leq m_i \leq f(x)$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$. It follows that the rectangle with vertices $(x_{i-1}, 0)$, $(x_i, 0)$, (x_i, m_i) and (x_{i-1}, m_i) is contained in the region X_i. This rectangle has width $x_i - x_{i-1}$ and height m_i, and thus has area $m_i(x_i - x_{i-1})$. It follows that

$$m_i(x_i - x_{i-1}) \leq \mathrm{area}(X_i)$$

for all integers i between 1 and n. Summing these inequalities over i, we find that

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \mathrm{area}(X_i) = \mathrm{area}(X).$$

An analogous inequality holds for upper sums. For each integer i between x_{i-1} and x_i the region X_i of the plane $\mathbb{R}^2$ is contained within the rectangle with vertices $(x_{i-1}, 0)$, $(x_i, 0)$, (x_i, M_i) and (x_{i-1}, M_i). This rectangle has width $x_i - x_{i-1}$ and height M_i, and thus has area $M_i(x_i - x_{i-1})$. It follows that

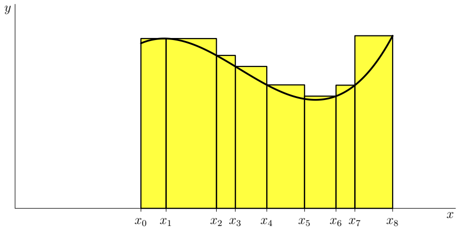
$$M_i(x_i - x_{i-1}) \geq \mathrm{area}(X_i)$$

for all integers i between 1 and n. Summing these inequalities over i, we find that

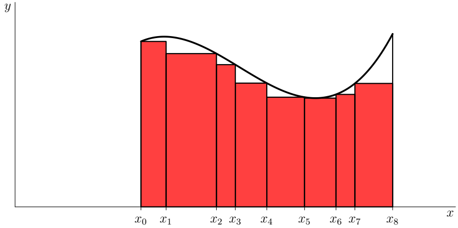
$$U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \geq \sum_{i=1}^{n} \mathrm{area}(X_i) = \mathrm{area}(X).$$

We conclude therefore that if the function f is non-negative on the interval $a, b]$, and if the region X "under the graph of the function" on the interval $[a, b]$ has a well-defined area, then

$$L(P, f) \leq \mathrm{area}(X) \leq U(P, f).$$



The upper sum $U(P, f)$



The lower sum $L(P, f)$

7.2. Upper and Lower Integrals and Integrability

Definition

Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U} \int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L} \int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

$$\begin{aligned} \mathcal{U} \int_a^b f(x)\,dx &= \inf \{ U(P, f) \mid P \text{ is a partition of } [a, b] \}, \\ \mathcal{L} \int_a^b f(x)\,dx &= \sup \{ L(P, f) \mid P \text{ is a partition of } [a, b] \}. \end{aligned}$$

The definition of upper and lower integrals thus requires that $\mathcal{U} \int_a^b f(x)\,dx$ be the infimum of the values of $U(P, f)$ and that $\mathcal{L} \int_a^b f(x)\,dx$ be the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$.

Remark

Let us consider how the lower and upper Riemann integrals of a bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ are related to the notion of the area "under the graph of the function f" on the interval a, in the case where the function f is non-negative on the interval $[a,b]$. Thus suppose that the region X has a well-defined area $\mathrm{area}(X)$, where

$$X = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}.$$

We have already shown that

$$L(P,f) \leq \mathrm{area}(X) \leq U(P,f)$$

for all partitions P of the interval $[a,b]$. It follows that $\mathrm{area}(X)$ is an upper bound on all the lower sums determined by all the partitions P of $[a,b]$. It is therefore not less than the least upper bound on all these lower sums. Therefore

$$\mathcal{L}\int_a^b f(x)\,dx \le \mathrm{area}(X),$$

An analogous argument shows that

$$\mathcal{U}\int_a^b f(x)\,dx \ge \mathrm{area}(X).$$

Thus if the region X has a well-defined area, then that area must satisfy the inequalities

$$\mathcal{L}\int_a^b f(x)\,dx \le \mathrm{area}(X) \le \mathcal{U}\int_a^b f(x)\,dx.$$

Definition

A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b, a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a, b]$, and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P, f) \leq L(P, g)$ and $U(P, f) \leq U(P, g)$ for all partitions P of $[a, b]$.