MA1S11—Calculus Portion
School of Mathematics, Trinity College
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6.5. Derivatives of Trigonometrical Functions

Lemma 6.10

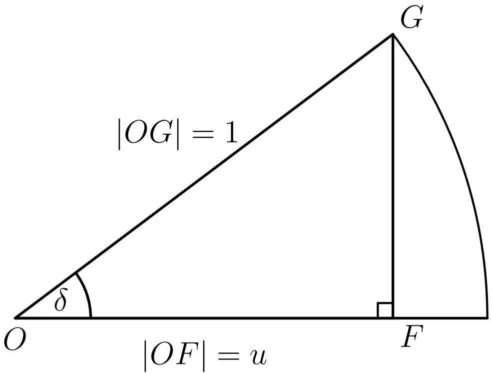
Let ε be a positive real number. Then there exists some positive real number δ satisfying $0 < \delta < \frac{1}{2}\pi$ with the property that $1 - \varepsilon < \cos\theta < 1$ whenever $0 < \theta < \delta$.

Proof

Choose a real number u satisfying $0 < u < 1$ for which $1 - \varepsilon \leq u$. Let a right-angled triangle OFG be constructed so that the angle at F is a right angle, $|OF| = u$ and $|FG| = \sqrt{1 - u^2}$, and let δ be the angle of this triangle at the vertex O. Then $|OG|^2 = |OF|^2 + |FG|^2 = 1$, and therefore $u = \cos\delta$. It follows that if θ is a positive real number satisfying $0 < \theta < \delta$ then

$$1 - \varepsilon \leq \cos\delta < \cos\theta < 1.$$

The result follows. ∎



G
$|OG| = 1$
δ
O
F
$|OF| = u$

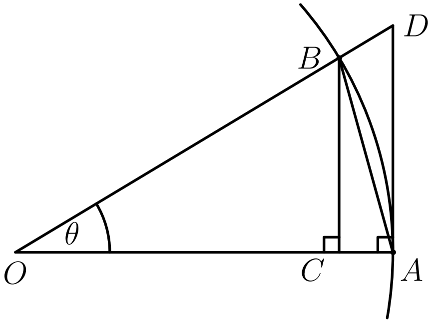
Proposition 6.11

Let $\sin\colon \mathbb{R} \to \mathbb{R}$ *be the sine function whose value* $\sin\theta$, *for a given real number* θ *is the sine of an angle of* θ *radians. Then*

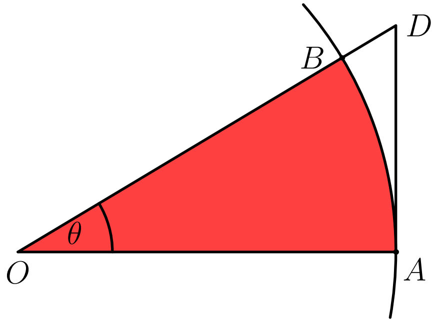
$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Proof

Let a circle of unit radius pass through points A and B, so that the angle θ in radians between the line segements OA and OB at the centre O of the circle satisfies the inequalities $0 < \theta < \frac{1}{2}\pi$. Let C be the point on the line segment OA for which the angle OCB is a right angle, and let the line OB be produced to the point D determined so that the angle OAD is a right angle.



The sector OAB of the unit circle is by definition the region bounded by the arc AB of the circle and the radii OA and OB. Now the area of a sector of a circle subtending at the centre an angle of θ radians is equal to the area of the circle multiplied by $\dfrac{\theta}{2\pi}$. But the area of a circle of unit radius is π. It follows that a sector of the unit circle subtending at the centre an angle of θ radians has area $\frac{1}{2}\theta$.

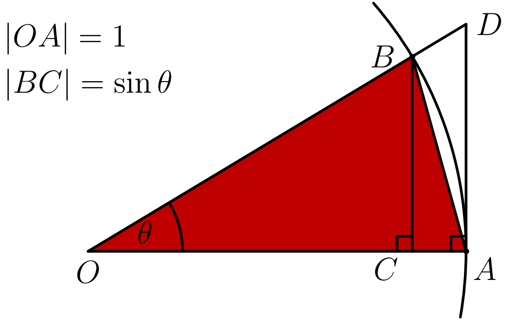


The the area of a triangle is half the base of the triangle multiplied by the height of the triangle. The base $|OA|$ and height $|BC|$ of the triangle AOB satisfy

$$|OA| = 1, \quad |BC| = \sin\theta.$$

It follows that

$$\text{area of triangle } OAB = \tfrac{1}{2} \times |OA| \times |BC| = \tfrac{1}{2}\sin\theta.$$

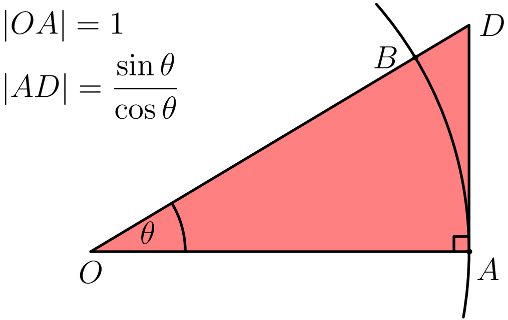


Also the base $|OA|$ and height $|AD|$ of the triangle AOD satisfy

$$|OA| = 1, \quad |AD| = \frac{\sin\theta}{\cos\theta}.$$

It follows that

$$\text{area of triangle } OAD = \tfrac{1}{2} \times |OA| \times |AD| = \frac{\sin\theta}{2\cos\theta}.$$



The results concerning areas just obtained can be summarized as follows:—

$$
\begin{aligned}
\text{area of triangle } OAB &= \tfrac{1}{2} \times |OA| \times |BC| \\
&= \tfrac{1}{2} \sin\theta, \\
\text{area of sector } OAB &= \frac{\theta}{2\pi} \times \pi = \tfrac{1}{2}\theta, \\
\text{area of triangle } OAD &= \tfrac{1}{2} \times |OA| \times |AD| \\
&= \tfrac{1}{2} \tan\theta = \frac{\sin\theta}{2\cos\theta}.
\end{aligned}
$$

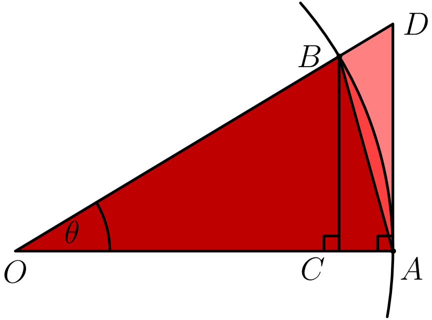
Moreover the triangle OAB is strictly contained in the sector OAB, which in turn is strictly contained in the triangle OAD. It follows that

$$\mathrm{area}(\triangle OAB) < \mathrm{area}(\mathrm{sector}\, OAB) < \mathrm{area}(\triangle OAD),$$

and thus

$$\tfrac{1}{2}\sin\theta < \tfrac{1}{2}\theta < \frac{\sin\theta}{2\cos\theta}$$

for all real numbers θ satisfying $0 < \theta < \frac{1}{2}\pi$.



Multiplying by 2, and then taking reciprocals, we find that

$$\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{\cos\theta}{\sin\theta}$$

for all real numbers θ satisfying $0 < \theta < \frac{1}{2}\pi$. If we then multiply by $\sin\theta$, we obtain the inequalities

$$\cos\theta < \frac{\sin\theta}{\theta} < 1,$$

for all real numbers θ satisfying $0 < \theta < \frac{1}{2}\pi$.

Now, given any positive real number ε, there exists some positive real number δ satisfying $0 < \delta < \frac{1}{2}\pi$ such that $1 - \varepsilon < \cos\theta < 1$ whenever $0 < \theta < \delta$ (see Lemma 6.10). But then

$$1 - \varepsilon < \frac{\sin\theta}{\theta} < 1$$

whenever $0 < \theta < \delta$. These inequalities also hold when $-\delta < \theta < 0$, because the value of $\dfrac{\sin\theta}{\theta}$ is unchanged on replacing θ by $-\theta$. It follows that $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$, as required. $\blacksquare$

Corollary 6.12

Let $\cos\colon \mathbb{R} \to \mathbb{R}$ *be the cosine function whose value* $\cos\theta$, *for a given real number* θ *is the cosine of an angle of* θ *radians. Then*

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$$

Proof

Basic trigonometrical identities ensure that

$$1 - \cos\theta = 2\sin^2 \tfrac{1}{2}\theta \quad \text{and} \quad \sin\theta = 2\sin\tfrac{1}{2}\theta\,\cos\tfrac{1}{2}\theta$$

for all real numbers θ (see Corollary 6.8 and Corollary 6.9). Therefore

$$\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta$$

for all real numbers θ. It follows that

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} = \lim_{\theta\to 0} \tan\tfrac{1}{2}\theta = 0,$$

and therefore

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} \times \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 0 \times 1 = 0,$$

as required. ∎

Proposition 6.13

The derivatives of the sine and cosine functions satisfy

$$\frac{d}{dx}(\sin x) = \cos x, \quad \textit{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

Proof

Limits of sums, differences and products of functions are the corresponding sums, differences and products of the limits of those functions, provided that those limits exist (see Proposition 4.17). Also

$$\sin(x+h) = \sin x\ \cos h + \cos x\ \sin h$$

and

$$\cos(x+h) = \cos x\ \cos h - \sin x\ \sin h$$

for all real numbers h (see Proposition 6.5). Applying these results, together with those of Proposition 6.11 and Corollary 6.12, we see that

$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\ \cos h + \cos x\ \sin h - \sin x}{h} \\
&= \cos x\ \lim_{h\to 0} \frac{\sin h}{h} - \sin x\ \lim_{h\to 0} \frac{1-\cos h}{h} \\
&= \cos x.
\end{aligned}
$$

Similarly

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x\ \cos h - \sin x\ \sin h - \cos x}{h} \\
&= -\sin x\ \lim_{h\to 0} \frac{\sin h}{h} - \cos x\ \lim_{h\to 0} \frac{1 - \cos h}{h} \\
&= -\sin x,
\end{aligned}$$

as required. ∎

Corollary 6.14

The derivative of the tangent function satisfies

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x.$$

Proof

Using the formulae for the derivatives of the sine and cosine functions (Proposition 6.13), together with the Quotient Rule for differentiation (Proposition 5.4) we find that

$$
\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{1}{\cos^2 x}\left(\frac{d}{dx}(\sin x)\cos x - \frac{d}{dx}(\cos x)\sin x\right) \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}
$$

as required. ■

6.6. The Inverse Tangent Function

Definition

The *inverse tangent function*

$$\arctan\colon \mathbb{R} \to (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$$

is defined so that, given any real number x, the quantity $\arctan x$ is the unique angle (specified in radian measure) for which $\tan(\arctan x)) = x$.

Thus the inverse tangent function is the unique function mapping the set $\mathbb{R}$ of real numbers into the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ with the property that $\tan(\arctan x)) = x$ for all real numbers x.

Lemma 6.15

The inverse tangent function

$$\text{arctan}\colon \mathbb{R} \to (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$$

is continuous.

Proof

Let s be a real number, and let $\beta = \arctan s$. Let some positive real number ε be given. Then real numbers α and γ can be chosen so that $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$, $-\frac{1}{2}\pi < \gamma < \frac{1}{2}\pi$, and

$$\beta - \varepsilon < \alpha < \beta < \gamma < \beta + \varepsilon.$$

Let $u = \tan\alpha$ and $v = \tan\gamma$. The tangent function is increasing. It follows that the inverse tangent function is also increasing. Thus if x is a real number satisfying $u < x < v$ then $\alpha < \arctan x < \gamma$. Let δ be the smaller of the positive numbers $v - s$ and $s - u$. if x is a real number satisfying $s - \delta < x < s + \delta$ then $u < x < v$. But then

$$\arctan s - \varepsilon < \alpha < \arctan x < \gamma < \arctan s + \varepsilon.$$

Thus the inverse tangent function arctan is continuous at s, as required. ∎

Proposition 6.16

The inverse tangent function arctan *is differentiable, and*

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

for all real numbers x.

Proof

Let s be a real number. Then there exists a real number β satisfying $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$ for which $\tan\beta = s$. Let $Q\colon (-\frac{1}{2}\pi, \frac{1}{2}\pi) \to \mathbb{R}$ be defined such that

$$Q(\theta) = \begin{cases} \dfrac{\tan\theta - \tan\beta}{\theta - \beta} & \text{if } \theta \neq \beta; \\ \dfrac{1}{\cos^2\beta} & \text{if } \theta = \beta. \end{cases}$$

Now the continuity of the tangent function, together with standard theorems on continuity, ensures that $Q(\theta)$ is a continuous function of θ when $\theta \neq \beta$. The function Q is also continuous at β because

$$\lim_{\theta\to\beta} Q(\theta) = \lim_{\theta\to\beta} \frac{\tan\theta - \tan\beta}{\theta - \beta} = \left.\frac{d}{d\theta}(\tan\theta)\right|_{\theta=\beta} = \frac{1}{\cos^2\beta} = Q(\beta)$$

(see Corollary 6.14). It follows that the function $Q(\theta)$ is continuous at θ for all values of θ satisfying $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

Now the inverse tangent function is continuous (Lemma 6.15), and compositions of continuous functions are continuous (Proposition 4.26). It follows that $Q(\arctan x)$ is a continuous function of x. Also $\tan\beta = s$. Therefore

$$\begin{aligned}\lim_{x\to s} Q(\arctan x) &= Q(\arctan s) = Q(\beta) = \frac{1}{\cos^2\beta} \\ &= 1 + \tan^2\beta = 1 + s^2.\end{aligned}$$

(see Proposition 6.4). But

$$\lim_{x\to s} Q(\arctan x) = \lim_{x\to s} \frac{x - s}{\arctan x - \arctan s}.$$

It follows that

$$\begin{aligned}\left.\frac{d}{dx}(\arctan x)\right|_{x=s} &= \lim_{x\to s}\frac{\arctan x - \arctan s}{x-s}\\ &= \lim_{x\to s}\frac{1}{Q(\arctan x)} = \frac{1}{1+s^2}.\end{aligned}$$

Thus the inverse tangent function is differentiable, and moreover its derivative at any real number x is equal to $\dfrac{1}{1+x^2}$, as required. ∎

6.7. The Inverse Sine and Cosine Functions

Definition

The *inverse sine function*

$$\arcsin\colon [-1,1] \to [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$$

is defined so that, given any real number x satisfying $-1 \leq x \leq 1$, the quantity $\arcsin x$ is the unique angle (specified in radian measure) satisfying the inequalities

$$-\tfrac{1}{2}\pi \leq \arcsin x \leq \tfrac{1}{2}\pi$$

for which $\sin(\arcsin x)) = x$.

Definition

The *inverse cosine function*

$$\arccos\colon [-1,1] \to [0,\pi]$$

is defined so that, given any real number x satisfying $-1 \leq x \leq 1$, the quantity $\arccos x$ is the unique angle (specified in radian measure) satisfying the inequalities

$$0 \leq \arcsin x \leq \pi$$

for which $\cos(\arccos x)) = x$.

The inverse sine and cosine functions are related by the identity

$$\arccos x = \frac{1}{2}\pi - \arcsin x$$

for all real numbers x satisfying $-1 \leq x \leq 1$.

We explore the relationship between the inverse sine and inverse tangent functions.

Lemma 6.17

The inverse sine function arcsin *is a differentiable function of x on the interval $(-1, 1)$ which satisfies the identity*

$$\arcsin x = \arctan\left(\frac{x}{\sqrt{1 - x^2}}\right).$$

when $-1 < x < 1$.

Proof

Let x be a real number satisfying $-1 < x < 1$ and let θ be the unique real number in the range $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ that satisfies $\sin\theta = x$. Then $\cos\theta = \sqrt{1 - x^2}$, and therefore

$$\tan\theta = \frac{x}{\sqrt{1 - x^2}}.$$

It follows that

$$\arcsin x = \theta = \arctan\left(\frac{x}{\sqrt{1 - x^2}}\right).$$

Now it follows from the Chain Rule (Proposition 5.5) that any composition of differentiable functions is differentiable. Therefore the inverse sine function is differentiable at x for all real numbers x satisfying $-1 < x < 1$, as required. $\blacksquare$

Proposition 6.18

The inverse sine function arcsin *satisfies*

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

for all real numbers x satisfying $-1 < x < 1$.

Proof

Differentiating the identity $\sin(\arcsin x) = x$ with respect to x using the Chain Rule (Proposition 5.5), we find that

$$\cos(\arcsin x)\,\frac{d}{dx}(\arcsin x) = 1.$$

Let $\theta = \arcsin x$. Then $x = \sin\theta$. It follows that

$$\cos(\arcsin x) = \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - x^2},$$

and therefore

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - x^2}},$$

as required. $\blacksquare$

Remark

The inverse sine and tangent functions are related by the identity

$$\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right).$$

when $-1 < x < 1$. Differentiating the right hand side of this identity using the Chain Rule, and using the result that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

(Proposition 6.16), we find that

$$\begin{aligned}
&\frac{d}{dx}\left(\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right) \\
&\qquad = \frac{1}{1+\dfrac{x^2}{1-x^2}}\,\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) \\
&\qquad = \frac{1-x^2}{(1-x^2)+x^2}\times\frac{(1-x^2)-\frac{1}{2}(-2x)x}{(1-x^2)^{\frac{3}{2}}} \\
&\qquad = (1-x^2)\times\frac{1}{(1-x^2)^{\frac{3}{2}}} \\
&\qquad = \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

This agrees with the formulae already found for the derivative of the inverse sine function (Proposition 6.18).

The inverse cosine function satisfies the identity

$$\arccos x = \frac{1}{2}\pi - \arcsin x.$$

It therefore follows from Proposition 6.18 that

$$\frac{d}{dx}(\arccos x) = -\frac{d}{dx}(\arcsin x) = -\frac{1}{\sqrt{1-x^2}}$$

for all real numbers x satisfying $-1 < x < 1$.