# MA1S11—Calculus Portion
# School of Mathematics, Trinity College
# Michaelmas Term 2016
# Lecture 22 (November 24, 2016)

David R. Wilkins

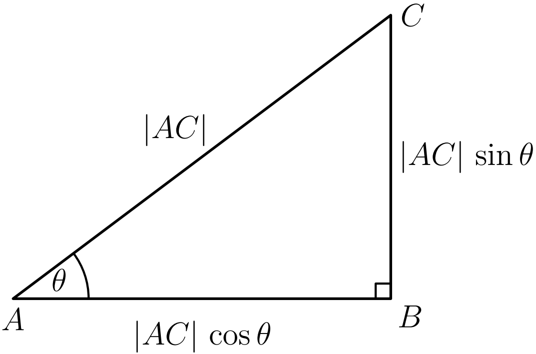
# 6. Trigonometric Functions and their Derivatives

## 6.1. Trigonometric Functions

There are six standard trigonometric functions. They are the *sine function* (sin), the *cosine function* (cos), the *tangent function* (tan), the *cotangent function* (cot), the *secant function* (sec) and the *cosecant function* (csc).

Angles will always be represented in the following discussion using *radian measure*. If one travels a distance $s$ around a circle of radius $r$, then the angle subtended by the starting and finishing positions at the centre of the circle is $s/r$ radians.

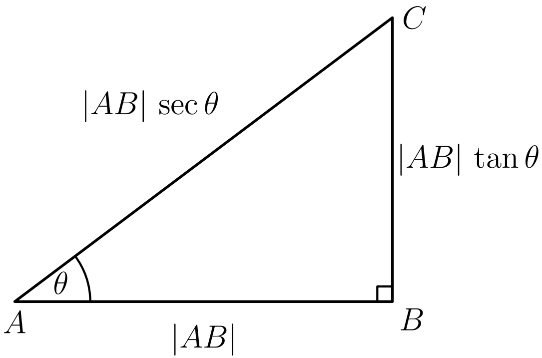
The standard trigonometrical functions represent ratios of sides of right-angled triangles, as indicated in the following diagrams.




In the above triangle $ABC$, in which the angle at the vertex $B$ is a right angle, the lengths $|AC|$, $|AB|$ and $|BC|$ satisfy the identities

$$|AB| = |AC| \cos\theta, \quad |AB| = |AC| \sin\theta,$$

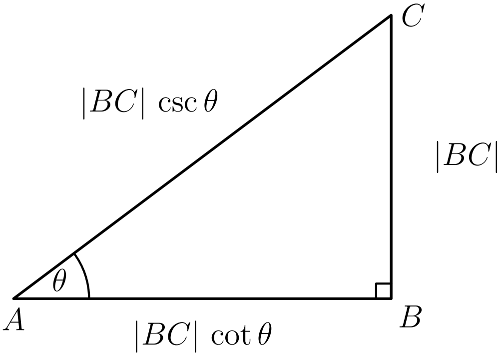
where $\theta$ denotes the angle of the triangle at the vertex $A$.




In the above triangle $ABC$, in which the angle at the vertex $B$ is a right angle, the lengths $|AC|$, $|AB|$ and $|BC|$ satisfy the identities

$$|BC| = |AB|\ \tan\theta, \quad |AC| = |AB|\ \sec\theta,$$

where $\theta$ denotes the angle of the triangle at the vertex $A$.




In the above triangle $ABC$, in which the angle at the vertex $B$ is a right angle, the lengths $|AC|$, $|AB|$ and $|BC|$ satisfy the identities

$$|AB| = |BC| \cot\theta, \quad |AC| = |BC| \csc\theta,$$

where $\theta$ denotes the angle of the triangle at the vertex $A$.

The identities described above that determine the ratios of the sides of a right angled triangle are summarized in the following proposition.

### Proposition 6.1

*Let $ABC$ be a triangle in which the angle at $B$ is a right angle, and let $\theta$ denote the angle at $A$. Then the lengths $|AB|$, $|BC|$ and $|AC|$ of the sides $AB$, $BC$ and $AC$ respectively satisfy the following identities:—*

$$|AB| = |AC| \cos\theta, \quad |BC| = |AC| \sin\theta;$$

$$|BC| = |AB| \tan\theta, \quad |AC| = |AB| \sec\theta;$$

$$|AB| = |BC| \cot\theta, \quad |AC| = |BC| \csc\theta.$$

The following trigonometrical formulae follow directly from the results stated in Proposition 6.1.

**Proposition 6.2**

*The tangent, cotangent, secant and cosecant functions are determined by the sine and cosine functions in accordance with the following identities:—*

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \quad \cot\theta = \frac{\cos\theta}{\sin\theta};$$

$$\sec\theta = \frac{1}{\cos\theta}; \quad \csc\theta = \frac{1}{\sin\theta}.$$

**Proposition 6.3**

*The sine and cosine functions are related by the following relationship, when angles are specified using radian measure:—*

$$\sin\theta = \cos(\tfrac{1}{2}\pi - \theta); \quad \cos\theta = \sin(\tfrac{1}{2}\pi - \theta).$$

**Proof**
The trigonometrical functions are determined by ratios of edges of a right angled triangle $ABC$ in which the angle $B$ is a right angle and the angle $A$ is $\theta$ radians. The angles of a triangle add up to two right angles, and two right angles are equal to $\pi$ in radian measure. Thus if $\angle B$ denotes the angle of the right-angled triangle $ABC$ then

$$\angle A + \angle B + \angle C = \pi,$$

and thus

$$\theta + \tfrac{1}{2}\pi + \angle C = \pi,$$

and therefore $C = \frac{1}{2}\pi - \theta$. The result then follows from the definitions of the sine and cosine functions. ∎

The $n$th powers of trigonometric functions are usually presented using the following traditional notation, in instances where $n$ is a positive integer:—

$$\sin^n\theta = (\sin\theta)^n, \quad \cos^n\theta = (\cos\theta)^n, \quad \tan^n\theta = (\tan\theta)^n, \quad \text{etc.}$$

**Proposition 6.4**

*The trigonometric functions satisfy the following identities:—*

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1; \\ 1 + \tan^2\theta &= \sec^2\theta; \\ 1 + \cot^2\theta &= \csc^2\theta; \end{aligned}$$

**Proof**

These identities follow from the definitions of the trigonometric functions on applying Pythagoras' Theorem. ∎

## 6.2. Periodicity of the Trigonometrical Functions

Suppose that a particle moves with speed $v$ around the circumference of a circle of radius $r$, where that circle is represented in Cartesian coordinates by the equation

$$x^2 + y^2 = r^2.$$

The centre of the circle is thus at the origin of the Cartesian coordinate system. We suppose that the particle travels in an anticlockwise direction and passes through the point $(r, 0)$ when $t = 0$. Then the particle will be at the point

$$\left( r\cos\frac{vt}{r}, r\sin\frac{vt}{r} \right).$$

at time $t$. The quantities $\sin\theta$ and $\cos\theta$ are defined for all real numbers $\theta$ so that the above formula for the position of the particle moving around the circumference of the circle at a constant speed remains valid for all times.

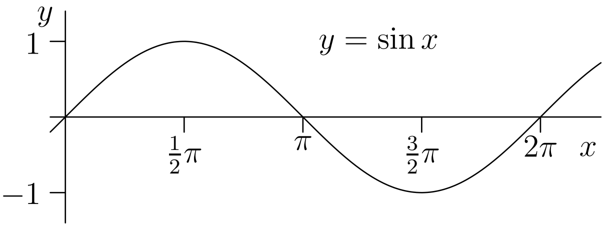
Now the particle moving round the circumference of the circle of radius $r$ with speed $v$ will complete each revolution in time $\dfrac{2\pi r}{v}$. Thus

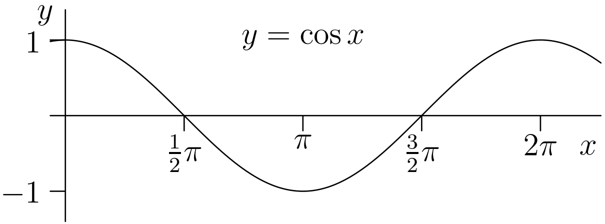
$$\cos(\theta + 2\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2\pi) = \sin\theta$$

for all real numbers $\theta$. It follows that

$$\cos(\theta + 2n\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2n\pi) = \sin\theta$$

for all real numbers $\theta$ and for all integers $n$. These equations express the *periodicity* of the sine and cosine functions.


$y$
$1$
$-1$
$y = \sin x$
$\frac{1}{2}\pi$
$\pi$
$\frac{3}{2}\pi$
$2\pi$
$x$


$y$
$1$
$-1$
$y = \cos x$
$\frac{1}{2}\pi$
$\pi$
$\frac{3}{2}\pi$
$2\pi$
$x$

## 6.3. Values of Trigonometric Functions at Particular Angles

The following table sets out the values of $\sin\theta$ and $\cos\theta$ for some angles $\theta$ that are multiples of $\frac{1}{2}\pi$:—

| $\theta$ | $-\pi$ | $-\frac{1}{2}\pi$ | $0$ | $\frac{1}{2}\pi$ | $\pi$ | $\frac{3}{2}\pi$ | $2\pi$ | $\frac{5}{2}\pi$ |
|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $-1$ | $0$ | $1$ | $0$ | $-1$ | $0$ | $1$ |
| $\cos\theta$ | $-1$ | $0$ | $1$ | $0$ | $-1$ | $0$ | $1$ | $0$ |

The following values of the sine and cosine functions can be derived using geometric arguments involving the use of Pythagoras' Theorem:—

| $\theta$ | $0$ | $\frac{1}{6}\pi$ | $\frac{1}{4}\pi$ | $\frac{1}{3}\pi$ | $\frac{1}{2}\pi$ |
|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| $\cos\theta$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ |

## 6.4. Addition Formulae satisfied by the Sine and Cosine Functions

We derive the standard addition formulae for trigonometric functions by considering the formulae that implement rotations of the plane about some origin chosen within that plane.

An anticlockwise rotation about the origin through an angle of $\theta$ radians sends a point $(x, y)$ of the plane to the point $(x', y')$, where

$$\left\{ \begin{array}{rcl} x' & = & x\cos\theta - y\sin\theta \\ y' & = & x\sin\theta + y\cos\theta \end{array} \right.$$

(This follows easily from the fact that such a rotation takes the point $(1, 0)$ to the point $(\cos\theta, \sin\theta)$ and takes the point $(0, 1)$ to the point $(-\sin\theta, \cos\theta)$.) An anticlockwise rotation about the origin through an angle of $\varphi$ radians then sends the point $(x', y')$ of the plane to the point $(x'', y'')$, where

$$\left\{ \begin{array}{rcl} x'' & = & x'\cos\varphi - y'\sin\varphi \\ y'' & = & x'\sin\varphi + y'\cos\varphi \end{array} \right.$$

Now an anticlockwise rotation about the origin through an angle of $\theta + \varphi$ radians sends the point $(x, y)$, of the plane to the point $(x'', y'')$, and thus

$$\begin{cases} x'' = x\cos(\theta+\varphi) - y\sin(\theta+\varphi) \\ y'' = x\sin(\theta+\varphi) + y\cos(\theta+\varphi) \end{cases}$$

But if we substitute the expressions for $x'$ and $y'$ in terms of $x$, $y$ and $\theta$ obtained previously into the above equation, we find that

$$\begin{cases} x'' = x(\cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi) - y(\sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi) \\ y'' = x(\sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi) + y(\cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi) \end{cases}$$

On comparing equations, we see that

$$\cos(\theta+\varphi) = \cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi,$$

and

$$\sin(\theta+\varphi) = \sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi.$$

On replacing $\varphi$ by $-\varphi$, and noting that $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -\sin\varphi$, we find that

$$\cos(\theta - \varphi) = \cos\theta\,\cos\varphi + \sin\theta\,\sin\varphi,$$

and

$$\sin(\theta - \varphi) = \sin\theta\,\cos\varphi - \cos\theta\,\sin\varphi.$$

We have therefore established the addition formulae for the sine and cosine functions stated in the following proposition.

**Proposition 6.5**

*The sine and cosine functions satisfy the following identities for all real numbers* $\theta$ *and* $\varphi$*:—*

$$\begin{aligned}
\sin(\theta+\varphi) &= \sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi;\\
\sin(\theta-\varphi) &= \sin\theta\,\cos\varphi - \cos\theta\,\sin\varphi;\\
\cos(\theta+\varphi) &= \cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi;\\
\cos(\theta-\varphi) &= \cos\theta\,\cos\varphi + \sin\theta\,\sin\varphi.
\end{aligned}$$

**Remark**

The equations describing how Cartesian coordinates of points of the plane transform under rotations about the origin may be written in matrix form as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Also equation (487) may be written

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

It follows from basic properties of matrix multiplication that

$$\begin{aligned}\begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \\ = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.\end{aligned}$$

Therefore

$$\begin{aligned} \cos(\theta+\varphi) &= \cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi \\ \sin(\theta+\varphi) &= \sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi. \end{aligned}$$

This provides an alternative derivation of the addition formulae stated in Proposition 6.5.

**Corollary 6.6**

*The sine and cosine functions satisfy the following identities for all real numbers $\theta$:—*

$$\begin{aligned} \sin(\theta + \tfrac{1}{2}\pi) &= \cos\theta, \\ \cos(\theta + \tfrac{1}{2}\pi) &= -\sin\theta, \\ \sin(\theta + \pi) &= -\sin\theta, \\ \cos(\theta + \pi) &= -\cos\theta, \end{aligned}$$

**Proof**

These results follow directly on applying Proposition 6.5 in view of the identities

$$\sin\tfrac{1}{2}\pi = 1, \quad \cos\tfrac{1}{2}\pi = 0, \quad \sin\pi = 0 \quad \text{and} \quad \cos\pi = -1. \qquad \blacksquare$$

The formulae stated in the following corollary follow directly from the addition formulae stated in Proposition 6.5 on adding and subtracting those addition formulae.

**Corollary 6.7**

*The sine and cosine functions satisfy the following identities for all real numbers $\theta$ and $\varphi$:—*

$$\begin{aligned}
\sin\theta\,\sin\varphi &= \tfrac{1}{2}(\cos(\theta-\varphi)-\cos(\theta+\varphi));\\
\cos\theta\,\cos\varphi &= \tfrac{1}{2}(\cos(\theta+\varphi)+\cos(\theta-\varphi));\\
\sin\theta\,\cos\varphi &= \tfrac{1}{2}(\sin(\theta+\varphi)+\sin(\theta-\varphi)).
\end{aligned}$$

**Corollary 6.8**

*The sine and cosine functions satisfy the following identities for all real numbers $\theta$:—*

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\,\cos\theta;\\
\cos 2\theta &= \cos^2\theta - \sin^2\theta\\
&= 2\cos^2\theta - 1\\
&= 1 - 2\sin^2\theta.
\end{aligned}$$

**Proof**

The formula for $\sin 2\theta$ and the first formula for $\cos 2\theta$ follow from the identities stated in Proposition 6.5 on setting $\varphi = \theta$ in the formulae for $\sin(\theta + \varphi)$ and $\cos(\theta + \varphi)$. The second and third formulae for $\cos 2\theta$ then follow on making use of the identity $\sin^2\theta + cos^2\theta = 1$. ∎

The following formulae then follow directly from those stated in Corollary 6.8.

**Corollary 6.9**

*The sine and cosine functions satisfy the following identities for all real numbers $\theta$:—*

$$\begin{aligned}\sin^2\theta &= \tfrac{1}{2}(1-\cos 2\theta);\\ \cos^2\theta &= \tfrac{1}{2}(1+\cos 2\theta).\end{aligned}$$