MA1S11—Calculus Portion
School of Mathematics, Trinity College
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David R. Wilkins

5.8. Rolle's Theorem

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function defined on a closed interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$ and

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

It then follows from the Extreme Value Theorem (Theorem 4.29) that there exist real numbers u and v in the interval $[a,b]$ such that

$$f(u) \leq f(x) \leq f(v)$$

for all real numbers x belonging to the interval $[a,b]$. The Extreme Value Theorem was stated *without proof* earlier in the course.

We now apply the Extreme Value Theorem, together with result that derivatives of differentiable functions are zero at local maxima and minima in the interior of the domain of the function (Proposition 5.7) in order to prove *Rolle's Theorem*

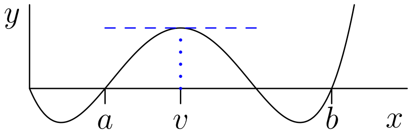
Theorem 5.8 (Rolle's Theorem)

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.

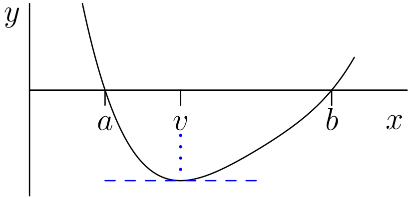
Proof

The function f is continuous on the closed bounded interval $[a, b]$. It therefore follows from the Extreme Value Theorem that there must exist real numbers u and v in the interval $[a, b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$ (see Theorem 4.29).

Suppose that $f(v) > f(a)$. Then $f(v) > f(b)$, because $f(a) = f(b)$. It follows that $v \neq a$ and $v \neq b$. But $a \leq v \leq b$. It must therefore be the case that $a < v < b$. Moreover $f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$. The function f thus attains a local maximum at v, where v is in the interior of the interval $[a, b]$, and therefore $f'(v) = 0$ (see Proposition 5.7). In this case therefore we can take $s = v$.



Next suppose that $f(u) < f(a)$. Then $f(u) < f(b)$, because $f(a) = f(b)$. It follows that $u \neq a$ and $u \neq b$. But $a \leq u \leq b$. It must therefore be the case that $a < u < b$. Moreover $f(x) \leq f(u)$ for all real numbers x satisfying $a \leq x \leq b$. The function f thus attains a local minimum at u, where u is in the interior of the interval $[a, b]$, and therefore $f'(u) = 0$ (see Proposition 5.7). In this case therefore we can take $s = u$.



The only remaining case to consider is the case when both u and v are endpoints of the interval $[a, b]$. In that case the function f is constant on $[a, b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. $\blacksquare$

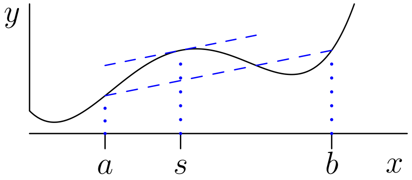
5.9. The Mean Value Theorem

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 5.9 (The Mean Value Theorem)

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Then there exists some real number s satisfying $a < s < b$ which has the property that

$$f(b) - f(a) = f'(s)(b-a).$$



Proof

Let $p\colon [a,b] \to \mathbb{R}$ be the function defined so that

$$p(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) = mx + k,$$

where

$$m = \frac{f(b) - f(a)}{b-a} \quad \text{and} \quad k = \frac{bf(a) - af(b)}{b-a}$$

Then $p(a) = f(a)$, $p(b) = f(b)$ and $p'(x) = m$ for all real numbers x satisfying $a \leq x \leq b$. (The equation $y = p(x)$ is then the equation of the line segment that joins the points $(a, f(a))$ and $(b, f(b))$ on the graph of f at $x = a$ and $x = b$.)

Next let $g\colon [a,b] \to \mathbb{R}$ be the function defined such that $g(x) = f(x) - p(x)$ for all real numbers x satisfying $a \leq x \leq b$. Then $g(a) = g(b) = 0$, because $f(a) = p(a)$ and $f(b) = p(b)$, and $g'(x) = f'(x) - m$ for all real numbers x satisfying $a \leq x \leq b$. It follows from Rolle's Theorem (Theorem 5.8) that there exists some real number s satisfying $a < s < b$ for which $g'(s) = 0$. But then

$$f'(s) = g'(s) + m = m = \frac{f(b) - f(a)}{b - a},$$

and thus $f(b) - f(a) = f'(s)(b - a)$, as required. ∎

5.10. Twice-Differentiable Functions

Definition

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a real number in the interior of D. The function f is said to be *twice-differentiable* at s if the derivative f' is defined and differentiable around s. The *second derivative* $f''(s)$ of a twice-differentiable function f at s is the value of the derivative of the derivative of f at s.

Let x be a real variable that ranges over a subset D of the set of real numbers, and let the dependent variable y be defined so that $y = f(x)$ for all values of x that belong to D, where $f\colon D \to \mathbb{R}$ is a twice-differentiable function on D. The first derivative $\dfrac{dy}{dx}$ of y with respect to x then satisfies

$$\frac{dy}{dx} = f'(x)$$

throughout D, and the second derivative $\dfrac{d^2y}{dx^2}$ of y with respect to x satisfies

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$$

throughout D.

5.11. The Second Derivative Test for Local Minima and Maxima

Proposition 5.10 (Second Derivative Test for Local Minimum)

Let $f: D \to \mathbb{R}$ be a twice-differentiable real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. Suppose that $f'(s) = 0$ and $f''(s) > 0$. Then the function f has a local minimum at s.

Proof
The first derivative f' of f satisfies

$$\lim_{x \to s} \frac{f'(x) - f'(s)}{x - s} = f''(s) > 0.$$

It follows that there exists some positive real number δ such that $x \in D$ and

$$\frac{f'(x) - f'(s)}{x - s} > \tfrac{1}{2} f''(s) > 0$$

whenever $s - \delta < x < s + \delta$ and $x \neq s$. But $f'(s) = 0$. It follows that

$$\frac{f'(x)}{x - s} > 0$$

whenever $s - \delta < x < s + \delta$ and $x \neq s$, and therefore $f'(x) > 0$ whenever $s < x < s + \delta$, and $f'(x) < 0$ whenever $s - \delta < x < s$.

Now it follows from the Mean Value Theorem (Theorem 5.9) that if x is a real number satisfying $s < x < s + \delta$ then there exists some real number v satisfying $s < v < x$ for which $f(x) - f(s) = f'(v)(x - s)$. But the derivative $f'(v)$ of f at v must then satisfy $f'(v) > 0$. It follows that $f(x) > f(s)$ whenever $s < x < s + \delta$.

It also follows from the Mean Value Theorem (Theorem 5.9) that if x is a real number satisfying $s - \delta < x < s$ then there exists some real number u satisfying $x < u < s$ for which $f(s) - f(x) = f'(u)(s - x)$. But the derivative $f'(u)$ of f at u must then satisfy $f'(v) < 0$. It follows that $f(x) > f(s)$ whenever $s - \delta < x < s$. We conclude from these results that the function f attains a local minimum at s, as required. ∎

Corollary 5.11 (Second Derivative Test for Local Maximum)

Let $f\colon D \to \mathbb{R}$ be a twice-differentiable real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. Suppose that $f'(s) = 0$ and $f''(s) < 0$. Then the function f has a local maximum at s.

Proof

This result follows immediately on applying Proposition 5.10 to the function $-f$. ∎

Let $f \colon D \to \mathbb{R}$ be a twice-differentiable real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. Suppose that $f'(s) = 0$. If $f''(s) > 0$ then the function f has a local minimum at s. If $f''(s) < 0$ then the function f has a local maximum at s. But if $f''(s) = 0$ then one is not in a position to draw any conclusion about whether there is a local minimum or maximum at s.

Example
Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = x^4$ for all real numbers x. Then $f'(0) = 0$ and $f''(0)$. The function f has a local minimum at zero.

Example
Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined so that $g(x) = -x^4$ for all real numbers x. Then $g'(0) = 0$ and $g''(0)$. The function g has a local maximum at zero.

Example
Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined so that $h(x) = x^3$ for all real numbers x. Then $h'(0) = 0$ and $h''(0)$. The function h has neither a local minimum nor a local maximum at zero.

5.12. Concavity and Points of Inflection

Let $f: D \to \mathbb{R}$ be a twice-differentiable function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that $f''(x) > 0$ for all $x \in I$. If u and v are real numbers belonging to the interval I that satisfy $u < v$ then from the Mean Value Theorem (Theorem 5.9) that there exists some real number s satisfying $u < s < v$ for which $f'(v) - f'(u) = f''(s)(v - u)$. But then $s \in I$, and therefore $f''(s) > 0$. It follows that $f'(u) < f'(v)$ for all real numbers u and v in the interval I. The graph of the function f thus becomes ever steeper as x increases through the interval I.

Now let x_1, x_2 and x_3 be real numbers belonging to the interval I that satisfy $x_1 < x_2 < x_3$. It follows from the Mean Value Theorem that there exist real numbers u and v satisfying $x_1 < u < x_2 < v < x_3$ such that

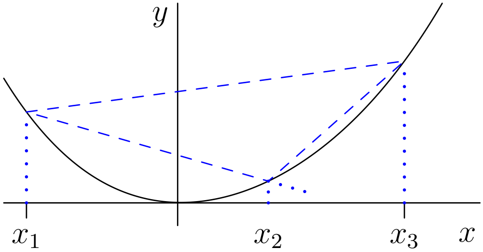
$$\frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(v) \quad \text{and} \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(u).$$

But $f'(u) < f'(v)$ because the second derivative of f is positive throughout the interval I. It follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

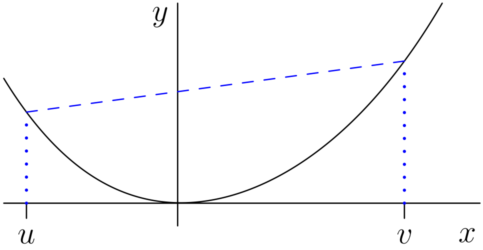
Thus the slope of the line segment joining the points $(x_2, f(x_2))$ and $(x_3, f(x_3))$ is greater than the slope of the line segment joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

It follows from this that the point $(x_3, f(x_3))$ lies above the line passing through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and therefore the point $(x_2, f(x_2))$ lies below the line joining the points $(x_1, f(x_1))$ and $(x_3, f(x_3))$. Moreover this argument applies for all values of x_2 that lie between x_1 and x_3. It follows that the graph of the function lies under the line segment joining the points $(x_1, f(x_1))$ and $(x_3, f(x_3))$.



Definition

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that, given real numbers u and v belonging to I that satisfy $u < v$, the line segment joining the point $(u, f(u))$ to the point $(v, f(v))$ lies above the graph of the function. Then the graph of the function is said to be *concave upwards* on the interval I.



The following result follows immediately from the preceding discussion.

Proposition 5.12

Let $f\colon D \to \mathbb{R}$ be a twice-differentiable function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that $f''(x) > 0$ for all $x \in I$. Then the graph of the function is concave upwards on I.

Definition

Let $f : D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that, given real numbers u and v belonging to I that satisfy $u < v$, the line segment joining the point $(u, f(u))$ to the point $(v, f(v))$ lies below the graph of the function. Then the graph of the function is said to be *concave downwards* on the interval I.

Corollary 5.13

Let $f : D \to \mathbb{R}$ be a twice-differentiable function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that $f''(x) < 0$ for all $x \in I$. Then the graph of the function is concave downwards on I.

Proof

The result follows immediately on applying Proposition 5.12 to the function $-f$. $\blacksquare$

Definition

Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. The point $(s, f(s))$ is said to be a *point of inflexion* of the graph of the function if s is common endpoint of an interval where the graph of the function is concave upwards and an interval where the graph of the function is concave downwards

Proposition 5.14

Let $f\colon D \to \mathbb{R}$ be a twice-differentiable function defined on a subset D of the set of real numbers, where the second derivative f'' is continuous on D, and let s be a point in the interior of D. Suppose that s determines a point of inflexion on the graph of the function f. Then $f''(s) = 0$.

Proof

If it were the case that $f''(s) > 0$ then the second derivative would be positive around s, and therefore the real number s would be in the interior of an interval on which the graph of the function is concave upwards (see Proposition 5.12). This is not possible. Therefore it cannot be the case that $f''(s) > 0$. An analogous argument shows that it cannot be the case that $f''(s) < 0$. (Indeed if the second derivative of f were negative at s then the second derivative of $-f$ would be positive at s, and we have shown that this is impossible.) Therefore $f''(s) = 0$, as required. ∎

5.13. The Newton-Raphson Method

Let $f\colon D \to \mathbb{R}$ be a differentiable function defined on a subset D of the set of real numbers. A *zero* (or *root*) of the function f is a real number x belonging to the domain of the function that satisfies the equation $f(x) = 0$.

Suppose we wish to locate zeros of the function f. There is an iterative method for locating zeros by successive approximations, generally known as the *Newton-Raphson Method*, which may in the appropriate circumstances determine the value of a zero of the function to a high degree of precision.

Let x_n be a real number in the domain D of the differentiable function $f\colon D \to \mathbb{R}$. Then the tangent line to the graph of the function f at $(x_n, f(x_n))$ satisfies the equation

$$y = f(x_n) + f'(x_n)(x - x_n),$$

where $f'(x_n)$ denotes the derivative of the function f at x_n. This tangent line crosses the x-axis at the point $(x_{n+1}, 0)$, where

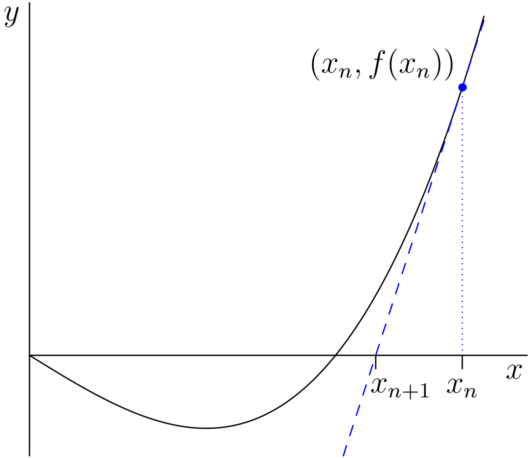
$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n).$$

Solving this equation for x_{n+1}, we find that

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}.$$

It follows that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$



y
$(x_n, f(x_n))$
x_{n+1}
x_n
x

The *Newton-Raphson* method for locating zeros of a differentiable function involves choosing an approximation x_1 to the zero, and then computing the sequence $x_1, x_2, x_3, x_4, \ldots$ of successive approximations to the zero so that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

for all positive integers n.

Example

Let $f(x) = x^3 - 2x$ for all real numbers x. Then $f'(x) = 3x^2 - 2$. We take $x_1 = 2$ as our initial approximation to a root of $f(x)$. Successive approximations are then determined by the Newton-Raphson method, so that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^3 - 2x}{3x^2 - 2}$$

for all natural numbers n. A computer-assisted calculation yields the following values for the successive approximations obtained:—

$$\begin{aligned}
x_1 &= 2.0,\\
x_2 &= 1.6,\\
x_3 &= 1.4422535211267606\ldots,\\
x_4 &= 1.415010636743953\ldots,\\
x_5 &= 1.4142142353546963\ldots,\\
x_6 &= 1.4142135623735754\ldots,\\
x_7 &= 1.4142135623730951\ldots,\\
x_8 &= 1.4142135623730951\ldots,
\end{aligned}$$