MA1S11—Calculus Portion
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5. Differential Calculus

5.1. Continuity of Differentiable Functions

Differentiable functions are continuous, as the following lemma shows.

Lemma 5.1

Let s be some real number, and let f be a differentiable real-valued functions defined throughout some neighbourhood of s. Then the function f is continuous at s, and thus $\lim_{x \to s} f(x) = f(s)$.

Proof

The function f satisfies the identity

$$f(x) = \frac{f(x) - f(s)}{x - s} \times (x - s) + f(s)$$

for all real numbers x satisfying $x \neq s$ that lie sufficiently close to s. Now limits of sums and products of functions are the sums and products of the respective limits where those limits are defined (see Proposition 4.17). It follows that

$$\begin{aligned} \lim_{x \to s} f(x) &= \lim_{x \to s} \left(\frac{f(x) - f(s)}{x - s} \right) \times \lim_{x \to s} (x - s) + f(s) \\ &= f'(s) \times 0 + f(s) = f(s). \end{aligned}$$

This ensures that the function f is continuous at s. (Proposition 4.21). The result follows. ∎

5.2. Derivatives of Sums and Differences of Functions

Proposition 5.2

Let s be some real number, and let f and g be real-valued functions defined throughout some neighbourhood of s. Suppose that the functions f and g are differentiable at s. Then $f + g$ and $f - g$ are differentiable at s, and

$$(f+g)'(s) = f'(s) + g'(s), \qquad (f-g)'(s) = f'(s) - g'(s).$$

Proof

Let x be a real number satisfying $x \neq s$ that is close enough to s to ensure that both $f(x)$ and $g(x)$ are defined at x. Now limits of sums and products of functions are the sums and products of the respective limits where those limits are defined (see Proposition 4.17). It follows that

$$\begin{aligned}
&\lim_{x \to s} \frac{(f+g)(x) - (f+g)(s)}{x - s} \\
&\qquad = \lim_{x \to s} \frac{f(x) - f(s)}{x - s} + \lim_{x \to s} \frac{g(x) - g(s)}{x - s} \\
&\qquad = f'(s) + g'(s).
\end{aligned}$$

Thus the function $f + g$ is differentiable at s, and $(f + g)'(s) = f'(s) + g'(s)$. An analogous proof shows that the function $f - g$ is also differentiable at s and $(f - g)'(s) = f'(s) - g'(s)$. ∎

5.3. The Product Rule

Proposition 5.3 (Product Rule)

Let s be some real number, and let f and g be differentiable real-valued functions defined throughout some neighbourhood of s. Let $f \cdot g$ denote the product function, defined so that $(f \cdot g)(x) = f(x)g(x)$ for all real numbers x for which both $f(x)$ and $g(x)$ are defined. Then the product function $f \cdot g$ is also differentiable at s, and

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s).$$

Proof

Let x be a real number satisfying $x \neq s$ that is close enough to s to ensure that both $f(x)$ and $g(x)$ are defined at x. Then

$$\begin{aligned} & \frac{f(x)g(x) - f(s)g(s)}{x - s} \\ & \qquad = \quad \frac{f(x) - f(s)}{x - s} g(x) + f(s) \frac{g(x) - g(s)}{x - s}. \end{aligned}$$

Now $\lim_{x \to s} g(x) = g(s)$ because the differentiable function g is necessarily continuous at s (see Lemma 5.1). Also limits of sums and products of functions are the sums and products of the respective limits where those limits are defined (see Proposition 4.17). It follows that

$$\begin{aligned}
&\lim_{x \to s} \frac{f(x)g(x) - f(s)g(s)}{x - s} \\
&\quad = \lim_{x \to s} \frac{f(x) - f(s)}{x - s} \lim_{x \to s} g(x) + f(s) \lim_{x \to s} \frac{g(x) - g(s)}{x - s} \\
&\quad = f'(s)g(s) + f(s)g'(s).
\end{aligned}$$

Thus the function $f \cdot g$ is differentiable at s, and

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s),$$

as required. ∎

5.4. The Quotient Rule

Proposition 5.4 (Quotient Rule)

Let s be some real number, and let f and g be differentiable real-valued functions defined throughout some neighbourhood of s, where $g(s) \neq 0$. Let f/g denote the product function, defined so that $(f/g)(x) = f(x)/g(x)$ for all real numbers x for which $f(x)$ and $g(x)$ are defined and $g(x) \neq 0$. Then the quotient function f/g is differentiable at s, and

$$(f/g)'(s) = \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2}.$$

Proof

Let x be a real number satisfying $x \neq s$ that is close enough to s to ensure that both $f(x)$ and $g(x)$ are defined at x and that $g(x) \neq 0$. Then

$$\begin{aligned} \frac{f(x)}{g(x)} - \frac{f(s)}{g(s)} &= \frac{f(x)g(s) - f(s)g(x)}{g(x)g(s)} \\ &= \frac{(f(x) - f(s))\, g(s) - f(s)\, (g(x) - g(s))}{g(s)g(x)}. \end{aligned}$$

Now $\lim_{x \to s} g(x) = g(s)$ because the differentiable function g is necessarily continuous at s (see Lemma 5.1). Also limits of sums, products and quotients of functions are the sums, products and quotients of the respective limits where those limits and quotients are defined (see Proposition 4.17). It follows that

$$
\begin{aligned}
&(f/g)'(s) \\
&= \lim_{x \to s} \frac{1}{x-s} \left(\frac{f(x)}{g(x)} - \frac{f(s)}{g(s)} \right) \\
&= \lim_{x \to s} \left(\frac{1}{g(x)g(s)} \right) \\
&\qquad \times \left(\lim_{x \to s} \frac{f(x)-f(s)}{x-s} g(s) - f(s) \lim_{x \to s} \frac{g(x)-g(s)}{x-s} \right) \\
&= \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2},
\end{aligned}
$$

as required. ∎

5.5. The Chain Rule

Proposition 5.5 (Chain Rule)

Let s be some real number, let f be a real-valued function defined throughout some neighbourhood of s, and let g be a real-valued function defined throughout some neighbourhood of $f(s)$. Suppose that the function f is differentiable at s, and the function g is differentiable at $f(s)$. Then the composition function $g \circ f$ is differentiable at s, and

$$(g \circ f)'(s) = g'(f(s))f'(s).$$

Proof

Let $r = f(s)$, and let

$$Q(y) = \begin{cases} \dfrac{g(y) - g(r)}{y - r} & \text{if } y \neq r; \\ g'(r) & \text{if } y = r. \end{cases}$$

for values of y around r. By considering separately the cases when $f(x) \neq f(s)$ and $f(x) = f(s)$, we see that

$$g(f(x)) - g(f(s)) = Q(f(x))\big(f(x) - f(s)\big).$$

Now the function Q is continuous at r, where $r = f(s)$, because

$$\lim_{y \to r} Q(r) = \lim_{y \to r} \frac{g(y) - g(r)}{y - r} = g'(r) = Q(r)$$

(see Proposition 4.21). Also the function f is continuous at s, because it is differentiable at s (see Lemma 5.1).

It follows that the composition function $Q \circ f$ is continuous at s (Proposition 4.26), and thus

$$\lim_{x \to s} Q(f(x)) = Q(f(s)) = g'(f(s))$$

(Proposition 4.21).

The limit of a product of functions is the product of the respective limits (see Proposition 4.17). Applying this result, we see that

$$\begin{aligned} (g \circ f)'(s) &= \lim_{x \to s} \frac{g(f(x)) - g(f(s))}{x - s} \\ &= \lim_{x \to s} Q(f(x)) \lim_{x \to s} \frac{f(x) - f(s)}{x - s} \\ &= g'(f(s)) f'(s). \end{aligned}$$

The result follows. ∎

5.6. Rules for Differentiation

We summarize the basic rules for differentiation, expressed in the traditional language of real variables.

We regard a *real variable* as a real number x whose value can vary over some set D that is a subset of the set of real numbers. We say that a real variable y is a *dependent variable*, that can be represented as a function of a real variable x, where x takes values in a subset D of the set of real numbers, if the dependence of y of x can be represented by an equation of the form $y = f(x)$, where $f\colon D \to \mathbb{R}$ is a real-valued function on the set D. We say that the dependent variable y is *differentiable* with respect to x if the function f that determines the dependence of y on x is a differentiable function. The derivative $\dfrac{dy}{dx}$ of y with respect to x is then the function whose value is equal to the derivative $f'(s)$ of the function f at $x = s$.

Proposition 5.6

Let x be a real variable, taking values in a subset D of the real numbers, and let y, u and v dependent variables, expressible as functions of the independent variable x, that are differentiable with respect to x. Then the following results are valid:—

(i) *if $y = c$, where c is a real constant, then* $\dfrac{dy}{dx} = 0$;

(ii) *if $y = cu$, where c is a real constant, then* $\dfrac{dy}{dx} = c\dfrac{du}{dx}$;

(iii) *if $y = u + v$ then* $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$;

(iv) *if $y = x^q$, where q is a rational number, then* $\dfrac{dy}{dx} = qx^{q-1}$;

(v) *(Product Rule)* if $y = uv$ then $\dfrac{dy}{dx} = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$;

(vi) *(Quotient Rule)* if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$;

(vii) *(Chain Rule)* if y is expressible as a differentiable function of u, where u in turn is expressible as a differentiable function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$.

Proof

Properties (i), (ii), (iii) follow directly from the definition of the derivative as a limit and from standard results concerning sums and products of limits (see Proposition 4.17). Property (v) is a restatement of Proposition 5.3. Property (vi) is a restatement of Proposition 5.4 Property (vii) is a restatement of Proposition 5.5. $\blacksquare$

5.7. Local Maxima and Minima of Differentiable Functions

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number. We say that s belongs to the *interior* of D if there exist real numbers u and v satisfying $u < s < v$ such that the set D contains all real numbers x satisfying $u < x < v$.

We recall that, given real numbers u and v satisfying $u < v$, the interval (u, v) is defined so that

$$(u, v) = \{x \in \mathbb{R} \mid u < x < b\}.$$

Every real number s belonging to the interval (u, v) is then in the interior of (u, v). And a real number s is in the interior of a subset D of the set $\mathbb{R}$ of real numbers if and only if there exist real numbers u and v for which $u < s < v$ and $(u, v) \subset D$.

Remark

It may be helpful to contemplate the definition of the interior of a set D of real numbers as follows: a real number s belonging to D is in the interior of D if and only if if it is completely surrounded by real numbers belonging to D. The formal definition merely makes precise what is meant by saying that s is "completely surrounded" by real numbers belonging to D.

For example, consider the (important) case in which $D = [a, b]$, where a and b are real numbers satisfying $a < b$ and

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

The endpoints a and b of this interval are not completely surrounded by points of the interval. But those real numbers s that satisfy $a < s < b$ *are* completely surrounded by points of the interval $[a, b]$, and they belong to the interior of $[a, b]$, where that interior is defined in accordance with the formal definition given above.

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers. The function f has a local minimum at s, where $s \in D$, if and only if there exists some positive real number δ such that $f(x) \geq f(s)$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$. Similarly the function f has a local maximum at s, where $s \in D$, if and only if there exists some positive real number δ such that $f(x) \leq f(s)$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$.

(These definitions are to be found in Subsection 3.8 of the course notes.)

Proposition 5.7

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. Suppose that the function f has a local maximum or a local minimum at s, and that the function f is differentiable at s. Then $f'(s) = 0$.

Proof

Suppose that the function f attains a local minimum at s, where the real number s belongs to the interior of the set D. Suppose also that the function f is differentiable at s with derivative $f'(s)$. Then

$$f'(s) = \lim_{x \to s} \frac{f(x) - f(s)}{x - s} = \lim_{x \to s^+} \frac{f(x) - f(s)}{x - s}.$$

for all real numbers x greater than s that lie sufficiently close to s. But $f(x) \geq f(s)$ for all real numbers x that lie sufficiently close to s. It follows that

$$\frac{f(x) - f(s)}{x - s} \geq 0$$

for all real numbers x satisfying $x > s$ that lie sufficiently close to s. It follows that

$$\lim_{x \to s^+} \frac{f(x) - f(s)}{x - s} \geq 0$$

(see Proposition 4.18). It follows that $f'(s) \geq 0$.

Similarly

$$\frac{f(x) - f(s)}{x - s} \leq 0$$

for all real numbers x satisfying $x < s$ that lie sufficiently close to s. It follows that

$$f'(s) = \lim_{x \to s} \frac{f(x) - f(s)}{x - s} = \lim_{x \to s^-} \frac{f(x) - f(s)}{x - s} \leq 0.$$

Thus $f'(s) \geq 0$ and $f'(s) \leq 0$, and therefore $f'(s) = 0$.

Next suppose that the function f attains a local maximum at s, where s belongs to the interior of D and the function f is differentiable at s. Then the function $-f$ attains a local minumum at s, and therefore the derivative $-f'(s)$ of the function $-f$ at s is equal to zero. Thus $f'(s) = 0$. This completes the proof. ∎

Example

Let

$$f(x) = 20x^{\frac{9}{4}} - 288x^{\frac{5}{4}} + 2700x^{\frac{1}{4}}.$$

for all real numbers x belonging to the interval $[1, 6]$, where

$$[1, 6] = \{x \in \mathbb{R} \mid 1 \leq x \leq 6\}.$$

Differentiating, we find that

$$f'(x) = 45x^{\frac{5}{4}} - 360x^{\frac{1}{4}} + 675x^{-\frac{3}{4}}$$

for all real numbers x belonging to the interval $[1, 6]$. Now

$$f'(x) = 45x^{-\frac{3}{4}}(x^2 - 8x + 15)$$

for all $x \in [1, 6]$. The derivative $f'(x)$ must be zero at any local maxima or minima in the interior of the interval $[1, 6]$. Now if $1 \leq x \leq 6$, and if $f'(x) = 0$, then either $x = 3$ or else $x = 5$, because 3 and 5 are the roots of the quadratic polynomial $x^2 - 8x + 15$.

Moreover the behaviour of this quadratic polynomial shows that $f'(x) > 0$ when $1 \leq x < 3$ and when $5 < x \leq 6$, and $f'(x) < 0$ when $3 < x < 5$. It follows that the function f is increasing on the intervals $[1, 3]$ and $[5, 6]$, but is decreasing on the interval $[3, 5]$. It follows that the function f attains a local maximum when $x = 3$, and attains a local minimum when $x = 5$. Calculating the values of $f(x)$ when x takes the values 1, 3, 5 and 6, we find that

$$f(1) = 2342, \quad f(3) = 2653.2052\ldots,$$

$$f(5) = 2631.8139\ldots, \quad f(6) = 2648.1231$$

to four decimal places. Applying the Intermediate Value Theorem (Theorem 4.28), we see that $f(x)$ takes on all real values between $f(1)$ and $f(3)$ as x increases from 1 to 3. It follows from the above calculations that the range of the function is the interval $[f(1), f(3)]$, where $f(1) = 2342$ and $f(3) = 2653.2052$ to four decimal places.