MA1S11—Calculus Portion
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4.14. Limits as the Variable Tends to Infinity

We now give the formal definition of the limit

$$\lim_{x \to +\infty} f(x)$$

of a real-valued function as the variable x “tends to $+\infty$”.

Let $f \colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of R. We say that $f(x)$ is defined *for all sufficiently large values of x* if there exists a real number A with the property that $x \in D$ and thus $f(x)$ is defined for all real numbers x that satisfy $x > A$.

Note that, in the definitions and proofs that follow, all “positive” real numbers are strictly greater than zero. (The terms “positive” and “strictly positive” are synonymous: the word “strictly” may occasionally precede the word “positive” on occasion to emphasize the requirement that the quantity in question be strictly greater than zero.)

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers let $f\colon D \to \mathbb{R}$ be a real-valued function on D that is defined for all sufficiently large values of the real variable x. The real number L is said to be the *limit* of $f(x)$, as x tends to $+\infty$ if and only if the following criterion is satisfied:—

> *given any positive real number ε, there exists some positive real number N such that*
>
> $$L - \varepsilon < f(x) < L + \varepsilon$$
>
> *for all real numbers x that satisfy $x > N$.*

In a situation where $f: D \to \mathbb{R}$ is a real-valued function, where s and L are a real numbers, and where L is the limit of $f(x)$ as x tends to $+\infty$, then we can denote this fact by writing

$$\lim_{x \to +\infty} f(x) = L.$$

The following proposition is useful in enabling us to deduce immediately standard properties of limits of functions as the variable tends to infinity.

Proposition 4.32

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let L be a real number. Suppose that there exists a real number A large enough to ensure that $x \in D$ for all real numbers x satisfying $x > A$. Then

$$\lim_{x \to +\infty} f(x) = L \quad \textit{if and only if} \quad \lim_{u \to 0^+} f\left(\frac{1}{u}\right) = L.$$

Proof

Suppose that $\lim_{x \to +\infty} f(x) = L$. Let some positive real number ε be given. Let N be a positive real number, and let $\delta = \dfrac{1}{N}$. Then

$$L - \varepsilon < f(x) < L + \varepsilon$$

for all real numbers x satisfying $x > N$ if and only if

$$L - \varepsilon < f\left(\frac{1}{u}\right) < L + \varepsilon$$

for all real numbers u satisfying $0 < u < \delta$. The result follows. ∎

The following proposition follows on combining the results of Proposition 4.32, Proposition 4.17 and Proposition 4.27.

Proposition 4.33

Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be continuous functions defined over a subset D of the set of real numbers, and let s be a real number. Suppose that $f(x)$ and $g(x)$ are defined for all sufficiently large values of the real variable x. Suppose also that the limits

$$\lim_{x \to +\infty} f(x) \quad \textit{and} \quad \lim_{x \to +\infty} g(x)$$

exist. Then

$$\begin{aligned}
\lim_{x \to +\infty} (f(x) + g(x)) &= \lim_{x \to +\infty} f(x) + \lim_{x \to +\infty} g(x), \\
\lim_{x \to +\infty} (f(x) - g(x)) &= \lim_{x \to +\infty} f(x) - \lim_{x \to +\infty} g(x), \\
\lim_{x \to +\infty} (f(x) g(x)) &= \lim_{x \to +\infty} f(x) \times \lim_{x \to +\infty} g(x).
\end{aligned}$$

Also

$$\lim_{x \to +\infty} h(f(x)) = h\left(\lim_{x \to +\infty} f(x)\right)$$

for all real-valued functions $h \colon E \to \mathbb{R}$ that are defined and continuous throughout some neighbourhood of $\lim_{x \to +\infty} f(x)$.

If moreover $g(x) \neq 0$ for all real numbers x satisfying $x > s$ that lie sufficiently close to s, and if $\lim_{x \to +\infty} g(x) \neq 0$ then

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to +\infty} f(x)}{\lim\limits_{x \to +\infty} g(x)}.$$

Remark

We have indicated how to apply the result of Proposition 4.32 in order to deduce the results stated in Proposition 4.33 from standard theorems that apply when a real variable approaches a limit point of its domain. The results of Proposition 4.33 can also be proved directly.

For example, suppose that

$$\lim_{x\to+\infty} f(x) = L_1 \quad \text{and} \quad \lim_{x\to+\infty} g(x) = L_2.$$

Let some positive real number ε be given. Then there exist positive real numbers N_1 and N_2 such that

$$L_1 - \tfrac{1}{2}\varepsilon < f(x) < L_1 + \tfrac{1}{2}\varepsilon$$

whenever $x > N_1$ and

$$L_2 - \tfrac{1}{2}\varepsilon < g(x) < L_2 + \tfrac{1}{2}\varepsilon$$

whenever $x > N_2$. Let N be the maximum of N_1 and N_2. If $x > N$ then

$$L_1 + L_2 - \varepsilon < f(x) + g(x) < L_1 + L_2 + \varepsilon$$

It follows that

$$\lim_{x\to+\infty}(f(x) + g(x)) = L_1 + L_2 = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x).$$

Example

We show that

$$\lim_{x\to+\infty} \sqrt{\frac{16x^6 - 8x^3 + 5}{x^6 - 6x^5 + 15x^4}}$$

exists and determine its value. Now

$$\frac{16x^6 - 8x^3 + 5}{x^6 - 6x^5 + 15} = \frac{16 - 8x^{-3} + 5x^{-5}}{1 - 6x^{-1} + 15x^{-2}}$$

for all positive real numbers x. Moreover $\lim\limits_{x\to+\infty} x^{-1} = 0$, and therefore

$$\lim_{x\to+\infty}(16-8x^{-3}+5x^{-5}) = 16 \quad \text{and} \quad \lim_{x\to+\infty}(1-6x^{-1}+15x^{-2}) = 1.$$

It follows that

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{16x^6-8x^3+5}{x^6-6x^5+15}\right) &= \lim_{x\to+\infty}\left(\frac{16-8x^{-3}+5x^{-5}}{1-6x^{-1}+15x^{-2}}\right)\\
&= \frac{\lim\limits_{x\to+\infty}(16-8x^{-3}+5x^{-5})}{\lim\limits_{x\to+\infty}(1-6x^{-1}+15x^{-2})}\\
&= \frac{16}{1}=16,
\end{aligned}$$

and therefore

$$\lim_{x\to+\infty}\sqrt{\frac{16x^6-8x^3+5}{x^6-6x^5+15x^4}}=\sqrt{16}=4.$$

We can define also the concept of the limit $\lim_{x \to -\infty} f(x)$ of a real-valued function $f\colon D \to \mathbb{R}$ as x "tends to $-\infty$".

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers let $f\colon D \to \mathbb{R}$ be a real-valued function on D that is defined for all sufficiently large values of $-x$. The real number L is said to be the *limit* of $f(x)$, as x tends to $-\infty$ if and only if the following criterion is satisfied:—

given any positive real number ε, there exists some positive real number N such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

for all real numbers x that satisfy $x < -N$.

In a situation where $f: D \to \mathbb{R}$ is a real-valued function, where s and L are a real numbers, and where L is the limit of $f(x)$ as x tends to $-\infty$, then we can denote this fact by writing

$$\lim_{x \to -\infty} f(x) = L.$$

The relevant definitions ensure that

$$\lim_{x \to -\infty} f(x) = L \quad \text{if and only if} \quad \lim_{x \to +\infty} f(-x) = L.$$

Properties of limits as $x \to -\infty$ therefore follow directly from the properties of corresponding limits as $x \to +\infty$.

4.15. Functions Increasing and Decreasing without Bound

Definition

let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a limit point of D. We say that $f(x)$ *increases without bound* as x tends to s, and write $f(x) \to +\infty$ as $x \to s$, if and only if the following criterion is satisfied:—

given any positive real number M, there exists some positive real number δ such that

$$f(x) > M$$

for all real numbers x that satisfy $0 < |x - s| < \delta$.

Lemma 4.34

let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a limit point of D. Suppose that $f(x) > 0$ for all $x \in D$. Then $f(x) \to +\infty$ as $x \to s$ if and only if

$$\lim_{x \to s} \frac{1}{f(x)} = 0.$$

Proof

Suppose that $f(x)$ increases without bound as $x \to s$. Let some positive real number ε be given. Then there exists some positive real number δ such that

$$f(x) > \frac{1}{\varepsilon}$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta$. But then

$$0 < \frac{1}{f(x)} < \varepsilon$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta$, and therefore

$$\lim_{x \to s} \frac{1}{f(x)} = 0.$$

Conversely suppose that

$$\lim_{x \to s} \frac{1}{f(x)} = 0,$$

where $f(x) > 0$ for all $x \in D$. Let some positive real number M be given. The formal definition of limits then ensures the existence of a positive real number δ such that

$$0 < \frac{1}{f(x)} < \frac{1}{M}$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta$. But then $f(x) > M$ for all real numbers x in D that satisfy $0 < |x - s| < \delta$, and thus $f(x)$ increases without bound as $x \to s$. ∎

Definition

let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a limit point of D. We say that $f(x)$ *decreases without bound* as x tends to s, and write $f(x) \to -\infty$ as $x \to s$, if and only if the following criterion is satisfied:—

> *given any positive real number M, there exists some positive real number δ such that*
>
> $$f(x) < -M$$
>
> *for all real numbers x that satisfy $0 < |x - s| < \delta$.*

The following lemma follows immediately from the formal definitions of what is meant by saying that $f(x) \to +\infty$ as $x \to s$ and $-f(x) \to -\infty$ as $x \to s$.

Lemma 4.35

let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a limit point of D. Then $f(x) \to +\infty$ as $x \to s$ if and only if $-f(x) \to -\infty$ as $x \to s$.

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f: D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that $f(x)$ is defined for all real numbers x satisfying $x > s$ that lie sufficient close to s. We say that $f(x)$ *increases without bound* as x tends to s from above, and write $f(x) \to +\infty$ as $x \to s^+$ if and only if the following criterion is satisfied:—

> *given any positive real number M, there exists some positive real number δ such that*
>
> $$f(x) > M$$
>
> *for all real numbers x that satisfy $s < x < s + \delta$.*

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that $f(x)$ is defined for all real numbers x satisfying $x < s$ that lie sufficient close to s. We say that $f(x)$ *increases without bound* as x tends to s from below, and write $f(x) \to +\infty$ as $x \to s^-$ if and only if the following criterion is satisfied:—

> *given any positive real number M, there exists some positive real number δ such that*
>
> $$f(x) > M$$
>
> *for all real numbers x that satisfy $s - \delta < x < s$.*

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that there exists a constant A such that $f(x)$ is defined for all real numbers x satisfying $x > A$. We say that $f(x)$ *increases without bound* as x increases without bound, and write $f(x) \to +\infty$ as $x \to +\infty$ if and only if the following criterion is satisfied:—

> *given any positive real number M, there exists a real number N such that $f(x) > M$ for all real numbers x that satisfy $x > N$.*

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f: D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that there exists a constant A such that $f(x)$ is defined for all real numbers x satisfying $x > A$. We say that $f(x)$ *decreases without bound* as x increases without bound, and write $f(x) \to -\infty$ as $x \to +\infty$ if and only if the following criterion is satisfied:—

> *given any positive real number M, there exists a positive real number N such that $f(x) < M$ for all real numbers x that satisfy $x > N$.*

The following result follows directly on comparing the relevant definitions.

Proposition 4.36

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that there exists a constant A such that $f(x)$ is defined for all real numbers x satisfying $x > A$. Then the following statements are equivalent:—

(i) $f(x) \to +\infty$ *as* $x \to +\infty$;

(ii) $-f(x) \to -\infty$ *as* $x \to +\infty$;

(iii) $f\left(\dfrac{1}{u}\right) \to +\infty$ *as* $u \to 0^+$;

(iv) $f(x) > 0$ for all sufficiently large real numbers x and $\displaystyle\lim_{x \to +\infty} \frac{1}{f(x)} = 0$;

(v) $f\left(\dfrac{1}{u}\right) > 0$ for all sufficiently small positive real numbers u and $\displaystyle\lim_{u \to 0^+} \frac{1}{f\left(\dfrac{1}{u}\right)} = 0$.

Example

Let

$$f(x) = \sqrt[3]{\frac{x^3 + x}{8x^2 - 32x + 64}}$$

for all positive real numbers x. We consider the behavour of $f(x)$ as the variable x increases without bound. Now

$$\begin{aligned} f(x) &= \sqrt[3]{x}\,\sqrt[3]{\frac{x^2 + 1}{8x^2 - 32x^2 + 64}} \\ &= \sqrt[3]{x}\,\sqrt[3]{\frac{1 + x^{-2}}{8 - 32x^{-1} + 64x^{-2}}}. \end{aligned}$$

Moreover

$$\lim_{x \to +\infty} (1 + x^{-2}) = \lim_{u \to 0^+} (1 + u) = 1$$

and

$$\lim_{x \to +\infty} (8 + 32x^{-1} + 64x^{-2}) = \lim_{u \to 0^+} (8 + 32u + 64u^2) = 8.$$

The limit of a quotient of functions is the quotient of the limits, where those limits exist and the denominator is everywhere non-zero and has non-zero limit (see Proposition 4.33). It follows that

$$\lim_{x \to +\infty} \left(\frac{1 + x^{-2}}{8 - 32x^{-1} + 64x^{-2}} \right) = \frac{1}{8}.$$

It then follows from the continuity of the cube root function that

$$\lim_{x \to +\infty} \sqrt[3]{\frac{1 + x^{-2}}{1 - 4x^{-1} + 8x^{-2}}} = \frac{1}{2}.$$

Now
$$\lim_{x\to+\infty}\frac{1}{\sqrt[3]{x}} = \lim_{u\to 0^+}\frac{1}{\sqrt[3]{u^{-1}}} = \lim_{u\to 0^+}\sqrt[3]{u} = 0.$$
It follows that
$$\begin{aligned}\lim_{x\to+\infty}\frac{1}{f(x)} &= \lim_{x\to+\infty}\frac{1}{\sqrt[3]{x}}\times\lim_{x\to+\infty}\frac{1}{\sqrt[3]{\dfrac{1+x^{-2}}{8-32x^{-1}+64x^{-2}}}}\\ &= 0.\end{aligned}$$
Moreover $f(x) > 0$ for all positive numbers x. On applying property (iv) listed in the statement of Proposition 4.36, we conclude that that $f(x)\to+\infty$ as $x\to+\infty$.