MA1S11—Calculus Portion
School of Mathematics, Trinity College
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4.11. The Intermediate Value Theorem

The following theorem is stated *without proof*.

Theorem 4.28 (The Intermediate Value Theorem)

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function defined on a closed interval $[a,b]$, and let c be a real number that is between $f(a)$ and $f(b)$ (so that either $f(a) \leq c \leq f(b)$ or else $f(a) \geq c \geq f(b)$). Then there exists a real number s satisfying $a \leq s \leq b$ for which $f(s) = c$.

4.12. The Extreme Value Theorem

The following theorem is also stated *without proof*.

Theorem 4.29 (The Extreme Value Theorem)

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function defined on a closed interval $[a,b]$. Then there exist real numbers u and v in the interval $[a,b]$ such that

$$f(u) \leq f(x) \leq f(v)$$

for all real numbers x belonging to the interval $[a,b]$.

4.13. One-Sided Limits

Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a real number. We say that $f(x)$ is defined *for all real numbers x satisfying $x > s$ that lie sufficiently close to s* if there exists some real number u satisfying $u > s$ such that $x \in D$ and thus $f(x)$ is defined for all real numbers x satisfying $s < x < u$.

Similarly we say that $f(x)$ is defined *for all real numbers x satisfying $x < s$ that lie sufficiently close to s* if there exists some real number u satisfying $u < s$ such that $x \in D$ and thus $f(x)$ is defined for all real numbers x satisfying $u < x < s$.

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s and L be a real numbers. Suppose that $f(x)$ is defined for all real numbers x satisfying $x > s$ that lie sufficiently close to s. The real number L is said to be the *limit* of $f(x)$, as x tends to s from above if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that*
>
> $$L - \varepsilon < f(x) < L + \varepsilon$$
>
> *for all real numbers x that satisfy $s < x < s + \delta$.*

In a situation where $f\colon D \to \mathbb{R}$ is a real-valued function, where s and L are a real numbers, and where L is the limit of $f(x)$ as x tends to s from above, then we can denote this fact by writing

$$\lim_{x \to s^+} f(x) = L.$$

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set of real numbers, and let s and L be real numbers. We suppose that $f(x)$ is defined for all real numbers x satisfying $x > s$ that lie sufficiently close to s. Then the real number L is the limit of $f(x)$ as x tends to s from above if and only if L is the limit of $f(x)$ as x tends to s in $D \cap [s, +\infty)$. Therefore all general results concerning limits can be applied to limits from above.

Thus let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions defined over a subset D of the set of real numbers, and let s be a real number. Suppose that $f(x)$ and $g(x)$ are defined for all real numbers x satisfying $x < s$ that lie sufficiently close to s. Suppose also that the limits

$$\lim_{x\to s^+} f(x) \quad \text{and} \quad \lim_{x\to s^+} g(x)$$

exist. It then follows from Proposition 4.17 that

$$\begin{aligned}
\lim_{x\to s^+} (f(x) + g(x)) &= \lim_{x\to s^+} f(x) + \lim_{x\to s^+} g(x),\\
\lim_{x\to s^+} (f(x) - g(x)) &= \lim_{x\to s^+} f(x) - \lim_{x\to s^+} g(x),\\
\lim_{x\to s^+} (f(x)g(x)) &= \lim_{x\to s^+} f(x) \times \lim_{x\to s^+} g(x),
\end{aligned}$$

Also it follows from Proposition 4.27 that

$$\lim_{x \to s^+} h(f(x)) = h\left(\lim_{x \to s^+} f(x)\right)$$

for all real-valued functions $h \colon E \to \mathbb{R}$ that are defined and continuous throughout some neighbourhood of $\lim_{x \to s^+} f(x)$.

If moreover $g(x) \neq 0$ for all real numbers x satisfying $x > s$ that lie sufficiently close to s, and if $\lim_{x \to s^+} g(x) \neq 0$ then

$$\lim_{x \to s^+} \frac{f(x)}{g(x)} = \frac{\lim_{x \to s^+} f(x)}{\lim_{x \to s^+} g(x)}.$$

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s and L be a real numbers. Suppose that $f(x)$ is defined for all real numbers x satisfying $x < s$ that lie sufficiently close to s. The real number L is said to be the *limit* of $f(x)$, as x tends to s from below if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that*
>
> $$L - \varepsilon < f(x) < L + \varepsilon$$
>
> *for all real numbers x that satisfy $s - \delta < x < s$.*

In a situation where $f\colon D \to \mathbb{R}$ is a real-valued function, where s and L are a real numbers, and where L is the limit of $f(x)$ as x tends to s from below, then we can denote this fact by writing

$$\lim_{x \to s^-} f(x) = L.$$

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set of real numbers, and let s and L be real numbers. We suppose that $f(x)$ is defined for all real numbers x satisfying $x < s$ that lie sufficiently close to s. Then the real number L is the limit of $f(x)$ as x tends to s from below if and only if L is the limit of $f(x)$ as x tends to s in $D \cap (-\infty, s]$. Therefore all general results concerning limits can be applied to limits from below.

Thus let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions defined over a subset D of the set of real numbers, and let s be a real number. Suppose that $f(x)$ and $g(x)$ are defined for all real numbers x satisfying $x < s$ that lie sufficiently close to s. Suppose also that the limits

$$\lim_{x \to s^-} f(x) \quad \text{and} \quad \lim_{x \to s^-} g(x)$$

exist. It then follows from Proposition 4.17 that

$$\begin{aligned}
\lim_{x \to s^-} (f(x) + g(x)) &= \lim_{x \to s^-} f(x) + \lim_{x \to s^-} g(x), \\
\lim_{x \to s^-} (f(x) - g(x)) &= \lim_{x \to s^-} f(x) - \lim_{x \to s^-} g(x), \\
\lim_{x \to s^-} (f(x)g(x)) &= \lim_{x \to s^-} f(x) \times \lim_{x \to s^-} g(x),
\end{aligned}$$

Also it follows from Proposition 4.27 that

$$\lim_{x \to s^-} h(f(x)) = h\left(\lim_{x \to s^-} f(x) \right)$$

for all real-valued functions $h\colon E \to \mathbb{R}$ that are defined and continuous throughout some neighbourhood of $\lim\limits_{x \to s^-} f(x)$.
If moreover $g(x) \neq 0$ for all real numbers x satisfying $x > s$ that lie sufficiently close to s, and if $\lim\limits_{x \to s^-} g(x) \neq 0$ then

$$\lim_{x \to s^-} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to s^-} f(x)}{\lim\limits_{x \to s^-} g(x)}.$$

Proposition 4.30

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f: D \to \mathbb{R}$ be a real-valued function on D, and let s and L be a real numbers. Suppose that $f(x)$ is defined for all real numbers x satisfying $x \neq s$ that lie sufficiently close to s. Then

$$\lim_{x \to s} f(x) = L$$

if and only if

$$\lim_{x \to s^+} f(x) = L \quad \text{and} \quad \lim_{x \to s^-} f(x) = L.$$

Proof

Suppose that

$$\lim_{x \to s} f(x) = L.$$

Let some positive real number ε be given. Then there exists some positive number δ such that $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $0 < |x - s| < \delta$. But then $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $s < x < s + \delta$ and $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $s - \delta < x < s$. It follows that

$$\lim_{x \to s^+} f(x) = L \quad \text{and} \quad \lim_{x \to s^-} f(x) = L.$$

Conversely suppose that

$$\lim_{x \to s^+} f(x) = L \quad \text{and} \quad \lim_{x \to s^-} f(x) = L.$$

Let some positive real number ε be given. Then there exist positive real numbers δ_1 and δ_2 such that $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $s < x < s + \delta_1$ and $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $s - \delta_2 < x < s$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $0 < |x - s| < \delta$. Thus

$$\lim_{x \to s} f(x) = L.$$

This completes the proof. ∎

Corollary 4.31

Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers. and let s be a real number. Suppose that $f(x)$ is defined for all real numbers x satisfying $x \neq s$ that lie sufficiently close to s. Then the function f is continuous at s if and only if

$$\lim_{x \to s^+} f(x) = \lim_{x \to s^-} f(x) = f(s).$$

Proof

The result follows immediately on combining the results of Proposition 4.21 Proposition 4.30. ∎

Example

We determine the unique value of the constant c for which the function $f_c\colon \mathbb{R} \to \mathbb{R}$ is continuous, where

$$f_c(x) = \begin{cases} x^2 + x & \text{if } x \leq 2; \\ 5x + c & \text{if } x > 2. \end{cases}$$

We also investigate whether the resulting function is differentiable at $x = 2$.

Now

$$\lim_{x \to 2^-} f_c(x) = \lim_{x \to 2} (x^2 + x) = 2^2 + 2 = 6 = f_c(2)$$

and

$$\lim_{x \to 2^+} f_c(x) = \lim_{x \to 2} (5x + c) = 5 \times 2 + c = 10 + c.$$

It follows that

$$\lim_{x \to s^-} f_c(x) = \lim_{x \to s^+} f_c(x) = f_c(2)$$

if and only if $c = -4$.

We now investigate the differentiability at 2 of $f\colon \mathbb{R} \to \mathbb{R}$, where

$$f(x) = f_{-4}(x) = \begin{cases} x^2 + x & \text{if } x \leq 2; \\ 5x - 4 & \text{if } x > 2. \end{cases}$$

Now

$$\begin{aligned} \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2^-} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2^-} \frac{(x + 3)(x - 2)}{x - 2} \\ &= \lim_{x \to 2^-} (x + 3) = 5, \end{aligned}$$

and

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{5x - 10}{x - 2} = 5.$$

Applying Proposition 4.30, we see that

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = 5.$$

Thus the function f is differentiable at 2, and $f'(2) = 5$.

Note that, in this example

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} \quad \text{and} \quad \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

are equal to the values at $x = 2$ of the derivatives of the functions $x \mapsto x^2 + x$ and $x \mapsto 5x - 4$ respectively.