MA1S11—Calculus Portion
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4.10. Continuity

The concept of *continuity* for functions of a real variable is defined formally as follows.

Definition

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set of real numbers, and let s be a real number belonging to D. The function f is *continuous* at s if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$.

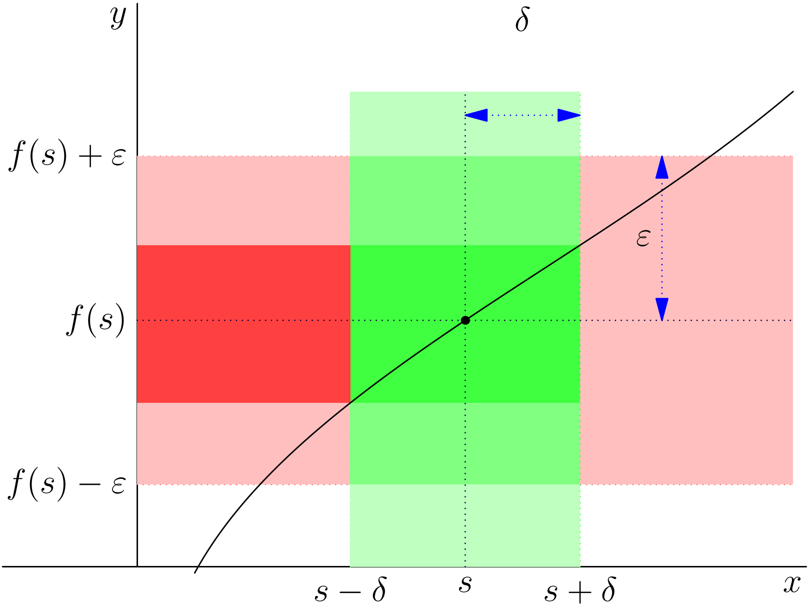
A real-valued function $f\colon D \to \mathbb{R}$ is said to be continuous on D if it is continuous at every real number belonging to D.

The definition of continuity can be expressed as follows: a real-valued function $f\colon D \to \mathbb{R}$ defined on a subset D of the set of real numbers is continuous at s, where $s \in D$, if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that

$$f(s) - \varepsilon < f(x) < f(s) + \varepsilon$$

for all real numbers x belonging to D that satisfy

$$s - \delta < x < s + \delta.$$


y
δ
$f(s)+\varepsilon$
ε
$f(s)$
$f(s)-\varepsilon$
$s-\delta$
s
$s+\delta$
x

Example

The function $f\colon \mathbb{R} \to \mathbb{R}$ defined such that $f(x) = x^3$ for all real numbers x is continuous. Indeed let s be a real number. Then

$$f(x) - f(s) = x^3 - s^3 = (x - s)(x^2 + xs + s^2).$$

Let $B = |s| + 1$. If x is a real number satisfying $s - 1 < x < s + 1$ then $-B \leq x \leq B$ and therefore

$$-3B^2 \leq x^2 + xs + s^2 \leq 3B^2.$$

It follows that

$$|f(x) - f(s)| \leq 3B^2 |x - s|$$

for all real numbers x satisfying $s - 1 < x < s + 1$.

Now let some positive real number ε be given. Then some positive real number δ can be chosen small enough to ensure that both $0 < \delta < 1$ and $\delta \leq \dfrac{\varepsilon}{3B^2}$. It then follows that if x is any real number satisfying $|x - s| < \delta$ then $-B \leq s - 1 < x < s + 1 \leq B$, and therefore

$$|f(x) - f(s)| \leq 3B^2|x - s| < 3B^2\delta \leq \varepsilon.$$

We have therefore verified that the formal definition of continuity is satisfied by the function f.

The definition of continuity is obviously closely related to the definition of limits. Indeed, examining definitions, we see that if $f \colon D \to \mathbb{R}$ is a real-valued function defined on a subset D of the set of real numbers, and if s is a real number belonging to D that is also a limit point of D, then the function f is continuous at s if and only if $\lim\limits_{x \to s} f(x) = f(s)$.

The following proposition states a necessary and sufficient condition for a function $f \colon D \to \mathbb{R}$ to be continuous on D.

Proposition 4.21

Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers. Then f is continuous on D if and only if $\lim_{x \to s} f(x) = f(s)$ for all real numbers s belonging to D that are limit points of D.

Proof

It follows from the definitions of limits and continuity that the function f is continuous at a real number s belonging to D that is also a limit point of D if and only if $\lim_{x \to s} f(x) = f(s)$.

If s is a real number belonging to D that is not a limit point of D then it follows from the definition of limit points that there exists some strictly positive real number δ for which

$$\{x \in D \mid |x - s| < \delta\} = \{s\}.$$

It then follows that $|f(x) - f(s)| = 0$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$, because the only real number x satisfying this inequality is s itself. It follows that the function f is continuous at any point of D that is not a limit point of D. The result follows. ∎

The following result follows immediately from Proposition 4.21 Proposition 4.17.

Proposition 4.22

Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be continuous functions defined over a subset D of the set of real numbers. Suppose that the functions f and g are continuous on D. Then the function $f + g$, $f - g$ and $f \cdot g$ are continuous on D, where $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$ and $(f \cdot g)(x) = f(x)g(x)$ for all real numbers x belonging to D. Moreover if the function g is non-zero throughout D then the function f/g is continuous on D, where $(f/g)(x) = f(x)/g(x)$ for all real numbers x belonging to D.

The following result follows immediately from Proposition 4.2 and Proposition 4.21. It can also be deduced through a straightforward application of Proposition 4.22

Proposition 4.23

All polynomial functions are continuous.

Example

We determine whether or not

$$\lim_{x \to 0} \frac{6x^2 + 8x^3 + 7x^5}{3x^2 + 8x^4 + x^7}$$

exists, and, if so, what is the value of the limit. Now the value of the limit of this expression at $x = 0$ is determined by the values of the expression for non-zero values of x. And

$$\frac{6x^2 + 8x^3 + 7x^5}{3x^2 + 8x^4 + x^7} = \frac{6 + 8x + 7x^3}{3 + 8x^2 + x^5}$$

when $x \neq 0$. Now the numerator and denominator of the fraction on the right hand side of the above equation are both polynomial functions. It follows that limit of these polynomial functions as x tends to zero is the value of the polynomials at $x = 0$ (see Proposition 4.2) Therefore

$$\lim_{x \to 0} (6 + 8x + 7x^3) = 6 \quad \text{and} \quad \lim_{x \to 0} (3 + 8x^2 + x^5) = 3.$$

It then follows from Proposition 4.22 that the limit of the given expression exists, and

$$\begin{aligned}\lim_{x\to 0}\frac{6x^2+8x^3+7x^5}{3x^2+8x^4+x^7} &= \lim_{x\to 0}\frac{6+8x+7x^3}{3+8x^2+x^5}\\ &= \frac{\lim\limits_{x\to 0}(6+8x+7x^3)}{\lim\limits_{x\to 0}(3+8x^2+x^5)}\\ &= \frac{6}{3}=2.\end{aligned}$$

Example

We now determine the value, if it exists, of

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x^2 + 2x - 15}.$$

In this case the numerator and denominator of the fraction are zero when $x = 3$. But both are quadratic polynomials which can be factored. Indeed

$$\frac{x^2 - 5x + 6}{x^2 + 2x - 15} = \frac{(x-3)(x-2)}{(x-3)(x+5)} = \frac{x-2}{x+5}$$

when $x \neq 3$. Moreover the numerator and denominator of the expression on the extreme right above are continuous functions of the variable x that are both non-zero when $x = 3$. It follows that

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x^2 + 2x - 15} = \lim_{x \to 3} \frac{(x-3)(x-2)}{(x-3)(x+5)} = \lim_{x \to 3} \frac{x-2}{x+5} = \frac{3-2}{3+5} = \frac{1}{8}.$$

Proposition 4.24

Let m and n be positive integers, and let s be a positive real number. Then

$$\lim_{x \to s} x^{\frac{m}{n}} = s^{\frac{m}{n}}$$

and

$$\lim_{x \to s} \frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x - s} = \frac{m}{n} s^{\frac{m}{n} - 1}.$$

Proof

Let s and x be positive real numbers, and let $u = s^{\frac{1}{n}}$ and $v = x^{\frac{1}{n}}$. Then

$$\begin{aligned}
\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} &= \frac{v^m - u^m}{v - u} = \frac{u^m - v^m}{u - v} = \sum_{j=0}^{m-1} u^{m-1-j} v^j \\
&= \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} x^{\frac{j}{n}}
\end{aligned}$$

(see Corollary 4.4).

The real number s is positive. We can therefore choose a positive real number A small enough to ensure that $0 < A^n < s$. If x satisfies $x > A^n$, and if $u = s^{\frac{1}{n}}$ and $v = x^{\frac{1}{n}}$ then $u > A$ and $v > A$. It follows that

$$\sum_{j=0}^{m-1} u^{m-1-j} v^j \geq \sum_{j=0}^{m-1} A^{m-1-j} A^j \geq \sum_{j=0}^{m-1} A^{m-1} = mA^{m-1}.$$

Thus

$$\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} \geq mA^{m-1}$$

whenever $s > A^n$ and $x > A^n$.

Applying this result in the case when $m = n$, we see that

$$\frac{x - s}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} \geq nA^{n-1},$$

and therefore

$$\frac{x^{\frac{1}{n}} - s^{\frac{1}{n}}}{x - s} \leq \frac{1}{nA^{n-1}}$$

for all real numbers x and s satisfying $s > A^n$ and $x > A^n$. It follows that

$$\frac{\left|x^{\frac{1}{n}} - s^{\frac{1}{n}}\right|}{|x - s|} = \frac{x^{\frac{1}{n}} - s^{\frac{1}{n}}}{x - s} \leq \frac{1}{nA^{n-1}}.$$

and thus

$$\left|x^{\frac{1}{n}} - s^{\frac{1}{n}}\right| \leq \frac{|x - s|}{nA^{n-1}},$$

for all real numbers x and s satisfying $s > A^n$ and $x > A^n$.

We claim that $\lim\limits_{x \to s} x^{\frac{1}{n}} = s^{\frac{1}{n}}$. Indeed let some positive real number ε be given, and δ be a positive real number chosen small enough to ensure that both $s - \delta \geq A^n$ and $0 < \delta \leq nA^{n-1}\varepsilon$. If x is a real number satisfying $0 < |x - s| < \delta$ then $x > s - \delta \geq A^n$ and therefore

$$|x^{\frac{1}{n}} - s^{\frac{1}{n}}| \leq \frac{|x - s|}{nA^{n-1}} < \frac{\delta}{nA^{n-1}} \leq \frac{nA^{n-1}\varepsilon}{nA^{n-1}} = \varepsilon.$$

Thus $\lim\limits_{x \to s} x^{\frac{1}{n}} = s^{\frac{1}{n}}$, as claimed.

Now the limit of a product of functions is the product of the limits of those functions (see Proposition 4.17). It follows that $\lim\limits_{x \to s} x^{\frac{m}{n}} = s^{\frac{m}{n}}$ for all positive integers m and n.

Now earlier in the proof we showed that if m and n are positive integers, and if x and s are positive real numbers then

$$\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} = \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} x^{\frac{j}{n}}.$$

Now the limit of a product of functions is the product of the limits of those functions, and the limit of a sum of functions is the sum of the limits of those functions (see Proposition 4.17). Applying these results, together with the Laws of Indices that apply to positive numbers raised to fractional powers (Proposition 1.14), we see that

$$\begin{aligned}
\lim_{x \to s} \left(\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} \right) &= \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} \lim_{x \to s} x^{\frac{j}{n}} \\
&= \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} \left(\lim_{x \to s} x^{\frac{1}{n}} \right)^{j} \\
&= \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} s^{\frac{j}{n}} \\
&= m s^{\frac{m-1}{n}}.
\end{aligned}$$

Applying this result with $m = n$, we find that

$$\lim_{x \to s} \left(\frac{x^{\frac{1}{n}} - s^{\frac{1}{n}}}{x - s} \right) = \frac{1}{\displaystyle\lim_{x \to s} \left(\frac{x - s}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} \right)} = \frac{1}{ns^{\frac{n-1}{n}}}.$$

It follows that

$$\begin{aligned} \lim_{x \to s} \left(\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x - s} \right) &= \lim_{x \to s} \left(\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} \right) \times \lim_{x \to s} \left(\frac{x^{\frac{1}{n}} - s^{\frac{1}{n}}}{x - s} \right) \\ &= ms^{\frac{m-1}{n}} \times \frac{1}{ns^{\frac{n-1}{n}}} = \frac{m}{n} s^{\frac{m-n}{n}} = \frac{m}{n} s^{\frac{m}{n} - 1}, \end{aligned}$$

as required. ∎

Corollary 4.25

Let q be a rational number. Then the function defined on the set of positive real numbers that maps each positive real number x to x^q is continuous and differentiable, and moreover

$$\frac{d}{dx}\left(x^q\right) = qx^{q-1}.$$

Proof

The result in the case $q > 0$ follows directly from Proposition 4.24. If $q = 0$ then the function f is constant, and the result is immediate.

Now suppose that $q < 0$. Let $p = -q$. Then

$$\frac{x^q - s^q}{x - s} = \frac{x^{-p} - s^{-p}}{x - s} = \frac{s^p - x^p}{x^p s^p (x - s)} = -\frac{x^p - s^p}{x^p s^p (x - s)}.$$

It follows from Proposition 4.24 and Proposition 4.17 that

$$\begin{aligned}
\lim_{x \to s} \frac{f(x) - f(s)}{x - s} &= \lim_{x \to s} \frac{x^q - s^q}{x - s} \\
&= -\lim_{x \to s} \frac{1}{x^p s^p} \times \lim_{x \to s} \frac{x^p - s^p}{x - s} \\
&= -\frac{1}{s^{2p}} \times p s^{p-1} = -p s^{-p-1} = q s^{q-1}.
\end{aligned}$$

The result follows. ∎

Remark

Let q be a rational number, and let $f\colon (0, +\infty) \to \mathbb{R}$ be the function defined so that $f(x) = x^q$ for all positive real numbers x.

Suppose that $q > 0$. Then the function f is increasing and its range is the set $(+, \infty)$ of positive real numbers. The continuity of f at a positive real number s can therefore be shown as follows.

Let ε be a positive real number, and let δ be the minimum of the positive numbers $(s^q + \varepsilon)^{\frac{1}{q}} - s$ and $s - (s^q - \varepsilon)^{\frac{1}{q}}$. If $s - \delta < x < s + \delta$ then $s^q - \varepsilon < x^q < s^q + \varepsilon$. Thus the function f is continuous.

A similar argument shows that the function f is continuous in the case when $q < 0$. In this case the function is decreasing.

Proposition 4.26

Let D and E be subsets of the set $\mathbb{R}$ of real numbers, let $f: \mathbb{R}$ and $g: E \to \mathbb{R}$ be continuous functions defined on D and E respectively, where $f(D) \subset E$, and let $g \circ f: D \to \mathbb{R}$ denote the composition of these functions, defined so that $(g \circ f)(x) = g(f(x))$ for all real numbers x belonging to D. Let s be a real number belonging to D. Suppose that the function f is continuous at s and that the function g is continuous at $f(s)$. Then the composition function $g \circ f$ is continuous at s.

Proof

Let some strictly positive real number ε be given. Then there exists some strictly positive real number η such that $|g(y) - g(f(s))| < \varepsilon$ for all real numbers y belonging to E that satisfy $|y - f(s)| < \eta$, because the function g is continuous at $f(x)$. But then there exists some strictly positive real number δ such that $|f(x) - f(s)| < \eta$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$. It follows that $|g(f(x)) - g(f(s))| < \varepsilon$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$, and thus the function $g \circ f$ is continuous at s, as required. $\blacksquare$

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = \sqrt{1 + 3x^2}$ for all real numbers x. The function $x \mapsto 1 + 3x^2$ is a polynomial function. It is therefore continuous on $\mathbb{R}$. (see Proposition 4.23). Also the function $u \mapsto \sqrt{u}$ is continuous on the set of positive real numbers (see Corollary 4.25). The function f is the composition of these two continuous functions. Therefore it is itself continuous. It then follows, for example, that

$$\lim_{x \to 1} \sqrt{1 + 3x^2} = \lim_{x \to 1} f(x) = f(1) = 2.$$

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined such that

$$f(x) = \frac{1 + \sqrt[5]{1 + x^2}}{\sqrt[3]{x^2 - 6x + 25}}$$

for all real numbers x. We show that this function is continuous. Now $x^2 - 6x + 25 > 0$ for all real numbers x. Polynomial functions are continuous (see Proposition 4.23). It follows that the function $x \mapsto x^2 - 6x + 25$ is continuous on $\mathbb{R}$. The function $u \mapsto \sqrt[3]{u}$ is continuous on the set of positive real numbers (see Corollary 4.25). Thus the function

$$x \mapsto \sqrt[3]{x^2 - 6x + 25}$$

is a composition of two continuous functions, and is thus itself a continuous function on $\mathbb{R}$ (see Proposition 4.26).

Similarly the function $x \mapsto \sqrt[5]{1 + x^2}$ is a continuous function on $\mathbb{R}$, and therefore the function

$$x \mapsto 1 + \sqrt[5]{1 + x^2}$$

is a continuous function on $\mathbb{R}$. It follows that the function f is a quotient of two continuous functions. It is therefore itself a continuous function (see Proposition 4.22).

Proposition 4.27

Let D and E be subsets of the set $\mathbb{R}$ of real numbers, let s be a limit point of D, let u be a point of E, let $f\colon D \to E$ be function satisfying $f(D) \subset E$, and let $g\colon E \to \mathbb{R}$ be a real-valued function on E. Suppose that

$$\lim_{x \to s} f(x) = u$$

and that the function g is continuous at u. Then

$$\lim_{x \to s} g(f(x)) = g(u).$$

Proof

Let some strictly positive real number ε be given. Then there exists some strictly positive real number η such that $|g(y) - g(u)| < \varepsilon$ for all real numbers y belonging to E that satisfy $|y - u| < \eta$, because the function g is continuous at u. But then there exists some positive real number δ such that $|f(x) - u| < \eta$ for all real numbers x belonging to D that satisfy $0 < |x - s| < \delta$. It follows that $|g(f(x)) - g(u)| < \varepsilon$ for all real numbers x belonging to D that satisfy $0 < |x - s| < \delta$, and thus

$$\lim_{x \to s} g(x) = g(u),$$

as required. ∎

Example

We now show that the limit

$$\lim_{x \to 0} \sqrt[3]{\frac{54x^4 - 108x^5 + 60x^6}{2x^4 - 12x^6 + 24x^8}},$$

exists, and determine the value of this limit. Now if $x \neq 0$ then

$$\frac{54x^4 - 108x^5 + 60x^6}{2x^4 + 12x^6 + 3x^{1}0} = \frac{54 - 108x + 60x^2}{2 + 12x^2 + 3x^6}.$$

Moreover

$$\lim_{x \to 0} (54 - 108x + 60x^2) = 54 \quad \text{and} \quad \lim_{x \to 0} (2 + 12x^2 + 3x^6) = 2.$$

It follows that

$$\begin{aligned}\lim_{x\to 0}\frac{54x^4-108x^5+60x^6}{2x^4+12x^6+3x^{10}} &= \lim_{x\to 0}\frac{54-108x+60x^2}{2+12x^2+3x^6}\\ &= \frac{\lim\limits_{x\to 0}(54-108x+60x^2)}{\lim\limits_{x\to 0}(2+12x^2+3x^6)}\\ &= \frac{54}{2}=27.\end{aligned}$$

The function defined on the set of positive real numbers that sends each positive real number u to $\sqrt[3]{u}$ is continuous. On applying Proposition 4.27, we see that

$$\lim_{x\to 0}\sqrt[3]{\frac{54x^4-108x^5+60x^6}{2x^4-12x^6+24x^8}}=\sqrt[3]{27}=3.$$