MA1S11—Calculus Portion
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4.9. Limits of Functions of a Real Variable

The following definition is the standard definition of limits of real-valued functions defined over subsets of the set $\mathbb{R}$ of real numbers.

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point belonging to D, and let L be a real number. The real number L is said to be the *limit* of $f(x)$, as x tends to s in D, if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - L| < \varepsilon$ for all real numbers x in D that satisfy $0 < |x - s| < \delta$.*

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point belonging to D, and let L be a real number. If L is the limit of $f(x)$ as x tends to s in D then we can denote this fact by writing $\lim\limits_{x \to s} f(x) = L$.

Note that the inequality $|f(x) - L| < \varepsilon$ is satisfied at x, where $x \in D$, if and only if

$$L - \varepsilon < f(x) < L + \varepsilon.$$

Also the inequality $0 < |x - s| < \delta$ is satisfied if and only if both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

Lemma 4.10

Let $f \colon D \to \mathbb{R}$ be a real-valued function defined on a subset $\mathbb{D}$ of $\mathbb{R}$, let s be a limit point of D, and let L be a real number. Suppose that $\lim_{x \to s} f(x) = L$. Then $\lim_{x \to s} (-f(x)) = -L$.

Proof

Let some positive real number ε be given. Then there exists some positive real number δ such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

whenever $0 < |x - s| < \delta$. Taking the negatives of the quantities satisfying the above inequalities we see that

$$-L - \varepsilon < -f(x) < -L + \varepsilon$$

whenever $0 < |x - s| < \delta$. We conclude that $\lim_{x \to s} (-f(x)) = -L$, as required. ∎

Lemma 4.11

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset $\mathbb{D}$ of $\mathbb{R}$, let s be a limit point of D, and let L and c be real numbers. Suppose that $\lim\limits_{x\to s} f(x) = L$. Then $\lim\limits_{x\to s}(f(x)+c) = L+c$.

Proof

Let some positive real number ε be given. Then there exists some positive real number δ such that

$$L-\varepsilon < f(x) < L+\varepsilon$$

whenever $0 < |x-s| < \delta$. Adding the constant c to all terms in these inequalities, we see that

$$L+c-\varepsilon < f(x)+c < L+c+\varepsilon$$

whenever $0 < |x-s| < \delta$. We conclude that $\lim\limits_{x\to s}(f(x)+c) = L+c$, as required. ∎

Lemma 4.12

Let $f : D \to \mathbb{R}$ be a real-valued function defined on a subset $\mathbb{D}$ of $\mathbb{R}$, let s be a limit point of D, and let L and M be real numbers where $M \neq 0$. Suppose that $\lim_{x \to s} f(x) = L$. Then $\lim_{x \to s} (Mf(x)) = ML$.

Proof

In view of Lemma 4.10 we need only consider the case when $M > 0$. Let some positive real number ε be given. Then some positive real number ε_0 can be chosen small enough to ensure that $M\varepsilon_0 < \varepsilon$. Then there exists some positive real number δ such that

$$L - \varepsilon_0 < f(x) < L + \varepsilon_0$$

whenever $0 < |x - s| < \delta$. But then

$$ML - \varepsilon < ML - M\varepsilon_0 < Mf(x) < ML + M\varepsilon_0 < ML + \varepsilon$$

whenever $0 < |x - s| < \delta$. We conclude that $\lim_{x \to s} (Mf(x)) = ML$, as required. ∎

Proposition 4.13

Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions on D, let s be a limit point of D, and let L and M be real numbers. Suppose that

$$\lim_{x \to s} f(x) = L$$

and

$$\lim_{x \to s} g(x) = M.$$

Then

$$\lim_{x \to s} (f(x) + g(x)) = L + M.$$

Proof

Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$L - \tfrac{1}{2}\varepsilon < f(x) < L + \tfrac{1}{2}\varepsilon$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta_1$ and

$$M - \tfrac{1}{2}\varepsilon < g(x) < M + \tfrac{1}{2}\varepsilon,$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if a real number x in D satisfies $0 < |x - s| < \delta$ then

$$L - \tfrac{1}{2}\varepsilon < f(x) < L + \tfrac{1}{2}\varepsilon$$

and

$$M - \tfrac{1}{2}\varepsilon < g(x) < M + \tfrac{1}{2}\varepsilon,$$

and therefore

$$L + M - \varepsilon < f(x) + g(x) < L + M + \varepsilon.$$

It follows that

$$\lim_{x \to s} (f(x) + g(x)) = L + M,$$

as required. ∎

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a limit point of D. Let $f\colon D \to \mathbb{R}$ be a real-valued function on D. We say that $f(x)$ *remains bounded* as x tends to s in D if there exist strictly positive constants C and δ such that $-C \leq f(x) \leq C$ for all real numbers x in D that satisfy $0 < |x - s| < \delta$.

Proposition 4.14

Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued function on some subset D of $\mathbb{R}$, and let s be a limit point of D. Suppose that $\lim\limits_{x \to s} f(x) = 0$. Suppose also that $g(x)$ remains bounded as x tends to s in D. Then

$$\lim_{x \to s} \Bigl(f(x) g(x) \Bigr) = 0.$$

Proof

Let some strictly positive real number ε be given. Then $g(x)$ remains bounded as x tends to s in D, and therefore positive constants C and δ_0 can be determined so that $-C \leq g(x) \leq C$ for all $x \in D$ satisfying $0 < |x - s| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(x)| < \varepsilon_0$ whenever $0 < |x - s| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |x - s| < \delta$ then $|g(x)| \leq C$ and $|f(x)| < \varepsilon_0$, and therefore

$$|f(x)g(x)| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.15

Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ be a function mapping D into $\mathbb{R}$, let $g\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of D, let L and M be real numbers. Suppose that

$$\lim_{x \to s} f(x) = L$$

and

$$\lim_{x \to s} g(x) = M.$$

Then

$$\lim_{x \to s} f(x)g(x) = LM.$$

Proof

The functions f and g satisfy the equation

$$f(x)g(x) = g(x)\Big(f(x) - L\Big) + (g(x) - M)L + LM,$$

where

$$\lim_{x \to s}\Big(f(x) - L\Big) = 0 \quad \text{and} \quad \lim_{x \to s}\Big(g(x) - M\Big) = 0.$$

Moreover there exists a strictly positive constant δ_0 such that

$$M - 1 < g(x) < M + 1$$

for all $x \in D$ satisfying $0 < |x - s| < \delta_0$. Thus the function g remains bounded as x tends to s in D. It now follows that

$$\lim_{x \to s}\left(g(x)(f(x) - L)\right) = 0$$

(see Proposition 4.14).

Similarly

$$\lim_{x \to s} (g(x) - M))\, L = 0.$$

It follows that

$$\begin{aligned} & \lim_{x \to s} (f(x) g(x)) \\ = \; & \lim_{x \to s} (g(x)(f(x) - L)) + \lim_{x \to s} \left(\Big(g(x) - M \Big) L \right) + LM \\ = \; & LM, \end{aligned}$$

as required. ∎

Lemma 4.16

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set $\mathbb{R}$ of real numbers, let s be a limit point of D, and let L be a real number. Suppose that $\lim_{x \to s} f(x) = L$, where $L \neq 0$.

$$\lim_{x \to s} \frac{1}{f(x)} = \frac{1}{L}.$$

Proof

We first prove the result in the case when $L > 0$. In this case we can choose a constant c such that $0 < c < L$. (For example, we could choose $c = \frac{1}{2}L$.) Now

$$\frac{1}{f(x)} - \frac{1}{L} = \frac{L - f(x)}{Lf(x)}.$$

It follows that

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| < \frac{1}{c^2}|f(x) - L|$$

whenever $f(x) \geq c$.

Now let some positive real number ε be given. Then a positive real number ε_0 can be found which is small enough to ensure that both $L - \varepsilon_0 \geq c$ and $0 < \varepsilon_0 \leq c^2\varepsilon$. (For example, we could take ε_0 to be the minimum of $L - c$ and ε/c^2.) Now $\lim\limits_{x \to s} f(x) = L$. It therefore follows that there exists some positive real number δ that is small enough to ensure that

$$|f(x) - L| < \varepsilon_0$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta$. It follows that if x is a real number in D that satisfies $0 < |x - s| < \delta$ then

$$c \leq L - \varepsilon_0 < f(x)$$

and

$$\left| \frac{1}{f(x)} - \frac{1}{L} \right| < \frac{1}{c^2} |f(x) - L| < \frac{\varepsilon_0}{c^2} \leq \varepsilon.$$

We conclude from this that if $f\colon D \to \mathbb{R}$ satisfies $\lim_{x\to s} f(x) = L$, where $L > 0$, then

$$\lim_{x\to s} \frac{1}{f(x)} = \frac{1}{L}.$$

Now suppose that satisfies $\lim_{x\to s} f(x) = L$, where $L < 0$, Then satisfies $\lim_{x\to s}(-f(x)) = -L$ where $-L > 0$ (see Lemma 4.10) and therefore

$$\lim_{x\to s} -\frac{1}{f(x)} = -\frac{1}{L}.$$

It follows that

$$\lim_{x\to s} \frac{1}{f(x)} = \frac{1}{L}$$

in this case also. This completes the proof. ∎

Proposition 4.17

Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions on D, and let s be a limit point of the set D. Suppose that $\lim_{x \to s} f(x)$ and $\lim_{x \to s} g(x)$ both exist. Then so do $\lim_{x \to s}(f(x) + g(x))$, $\lim_{x \to s}(f(x) - g(x))$ and $\lim_{x \to s}(f(x)g(x))$, and moreover

$$
\begin{aligned}
\lim_{x \to s}(f(x) + g(x)) &= \lim_{x \to s} f(x) + \lim_{x \to s} g(x), \\
\lim_{x \to s}(f(x) - g(x)) &= \lim_{x \to s} f(x) - \lim_{x \to s} g(x), \\
\lim_{x \to s}(f(x)g(x)) &= \lim_{x \to s} f(x) \times \lim_{x \to s} g(x),
\end{aligned}
$$

If moreover $g(x) \neq 0$ for all $x \in D$ and $\lim_{x \to s} g(x) \neq 0$ then

$$\lim_{x \to s} \frac{f(x)}{g(x)} = \frac{\lim_{x \to s} f(x)}{\lim_{x \to s} g(x)}.$$

Proof

It follows from Proposition 4.13 (applied in the case when the target space is one-dimensional) that

$$\lim_{x \to s} (f(x) + g(x)) = \lim_{x \to s} f(x) + \lim_{x \to s} g(x).$$

Replacing the function g by $-g$, we deduce that

$$\lim_{x \to s} (f(x) - g(x)) = \lim_{x \to s} f(x) - \lim_{x \to s} g(x).$$

It follows from Proposition 4.15 that

$$\lim_{x \to s} (f(x)g(x)) = \lim_{x \to s} f(x) \times \lim_{x \to s} g(x).$$

Now suppose that $g(x) \neq 0$ for all $x \in D$ and that $\lim\limits_{x \to s} g(x) \neq 0$. It follows from Lemma 4.16 that

$$\lim_{x \to s} \frac{1}{g(x)} = \frac{1}{\lim\limits_{x \to s} g(x)}.$$

It then follows from Proposition 4.15 that

$$\lim_{x \to s} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to s} f(x)}{\lim\limits_{x \to s} g(x)}.$$

This completes the proof. ∎

Proposition 4.18

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set $\mathbb{R}$ of real numbers, let s be a limit point of D, and let L be a real number. Suppose that $f(x) \geq 0$ for all real numbers x in D that satisfy $x \neq s$. Suppose also that $\lim_{x \to s} f(x) = L$. Then $L \geq 0$.

Proof

Suppose that it were the case that $L < 0$. Let $\varepsilon = -L$. Then $\varepsilon > 0$. Therefore there would exist some positive real number δ such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

for all real numbers x belonging to D that satisfy $0 < |x - s| < \delta$. But $\varepsilon = -L$. It would therefore follow that $f(x) < 0$ for all real numbers x belonging to D that satisfy $0 < |x - s| < \delta$. Moreover there exists at least elements of D that satisfy these inequalities because s is a limit point of D. Thus the hypothesis that $L < 0$ results in a contradiction. It follows that $\lim_{x \to s} f(x) \geq 0$, as required. ∎

Corollary 4.19

Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions defined over a subset D of the set $\mathbb{R}$ of real numbers, and let s be a limit point of D. Suppose that $f(x) \leq g(x)$ for all $x \in D$, and that $\lim_{x \to s} f(x)$ and $\lim_{x \to s} g(x)$ both exist. Then

$$\lim_{x \to s} f(x) \leq \lim_{x \to s} g(x).$$

Proof

The inequality $g(x) - f(x) \geq 0$ is satisfied for all $x \in D$. It follows from Proposition 4.17 and Proposition 4.18 that

$$\lim_{x \to s} g(x) - \lim_{x \to s} f(x) = \lim_{x \to s} (g(x) - f(x)) \geq 0,$$

and thus $\lim_{x \to s} f(x) \leq \lim_{x \to s} g(x)$, as required. ∎

Theorem 4.20 (Squeeze Theorem)

Let f, g and h be real-valued functions defined over a subset D of the set $\mathbb{R}$ of real numbers, let s be a limit point of D, and let L be a real number. Suppose that $f(x) \leq g(x) \leq h(x)$ for all real numbers x satisfying $x \neq s$ that belong to D. Suppose also that

$$\lim_{x \to s} f(x) = \lim_{x \to s} h(x) = L,$$

so that the real number L is the limit both of $f(x)$ and of $h(x)$ as x tends to s in D. Then

$$\lim_{x \to s} g(x) = L.$$

Proof

Let some positive real number ε be given. Then there exist positive real numbers δ_1 and δ_2 such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

whenever $0 < |x - s| < \delta_1$ and

$$L - \varepsilon < h(x) < L + \varepsilon$$

whenever $0 < |x - s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if a real number x belonging to D satisfies $0 < |x - s| < \delta$ then

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon,$$

and therefore $\lim_{x \to s} g(x) = L$, as required. ∎

Example

Let $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined so that

$$g(x) = 3\sqrt{|x|}\sin\left(\frac{\pi}{2x}\right)$$

for all non-zero real numbers x. Then

$$f(x) \leq g(x) \leq h(x)$$

for all non-zero real numbers x, where

$$f(x) = -3\sqrt{|x|} \quad \text{and} \quad h(x) = 3\sqrt{|x|}$$

for all real numbers x.

Now, given any positive real number ε, a positive real number δ could be chosen such that $3\sqrt{\delta} \leq \varepsilon$. For example, one could choose $\delta = \frac{1}{9}\varepsilon^2$. Then $-\varepsilon < f(x) \leq h(x) < \varepsilon$ for all real numbers x satisfying $0 < |x| < \delta$. We have thus shown that

$$\lim_{x \to 0} f(x) = 0 \quad \text{and} \quad \lim_{x \to 0} h(x) = 0.$$

It follows from the Squeeze Theorem (Theorem 4.20) that $\lim_{x \to 0} g(x) = 0$.