MA1S11—Calculus Portion
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4.7. Absolute Values of Real Numbers

Let x be a real number. The *absolute value* $|x|$ of x is defined so that

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0; \end{cases}$$

Lemma 4.9

Let x and y be real numbers. Then $|x + y| \leq |x| + |y|$ and $|xy| = |x|\,|y|$.

Proof

Let x and y be real numbers. Then

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|.$$

On adding inequalities, we find that

$$-(|x| + |y|) = -|x| - |y| \leq x + y \leq |x| + |y|,$$

and thus

$$x + y \leq |x| + |y| \quad \text{and} \quad -(x + y) \leq |x| + |y|.$$

Now the value of $|x + y|$ is equal to at least one of the numbers $x + y$ and $-(x + y)$. It follows that

$$|x + y| \leq |x| + |y|$$

for all real numbers x and y.

Next we note that $|x|\,|y|$ is the product of one or other of the numbers x and $-x$ with one or other of the numbers y and $-y$, and therefore its value is equal either to xy or to $-xy$. Because both $|x|\,|y|$ and $|xy|$ are non-negative, we conclude that $|xy| = |x|\,|y|$, as required. $\blacksquare$

Absolute values can be used in order to express some of the inequalities occurring in formal definitions of mathematical concepts such as limits, continuity and convergence in more compact form.

Consider the formal definition of limits. For simplicity we consider a real-valued function $f \colon \mathbb{R} \setminus \{s\} \to \mathbb{R}$ defined over the set $\mathbb{R} \setminus \{s\}$ of real numbers distinct from s. Let L be a real number. Then

$$\lim_{x \to s} f(x) = L$$

if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

for all real numbers x that satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

So let us first consider inequalities of the form

$$L - \varepsilon < y < L + \varepsilon,$$

where y, L and ε are real numbers and $\varepsilon > 0$. Subtracting L from both sides and using the definition of absolute values, we find that

$$\begin{aligned} & L - \varepsilon < y < L + \varepsilon \\ \iff \quad & -\varepsilon < y - L < \varepsilon \\ \iff \quad & |y - L| < \varepsilon \end{aligned}$$

(Here $\iff$ signifies "if and only if".)

Thus the condition on the values of the function f requiring that

$$L - \varepsilon < f(x) < L + \varepsilon$$

for values of real variable x satisfying appropriate constraints is completely equivalent to the condition that

$$|f(x) - L| < \varepsilon$$

for those values of x. Both forms of this condition express the fact that the quantity $f(x)$ is within a distance ε of the limiting value L.

This analysis shows that the function $f\colon \mathbb{R} \setminus \{s\} \to \mathbb{R}$ satisfies $\lim_{x \to s} f(x) = L$ if and only if, given any positive real number ε, there exists a positive real number δ such that

$$|f(x) - L| < \varepsilon$$

for all real numbers x that satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

(Note that, according to standard definitions, positive real numbers are required to be non-zero. Thus the terms "positive" and "strictly positive" are synonymous.)

Next we examine the constraints on the real variable x. This is required to satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

Now, from what has already been shown, we see that

$$s - \delta < x < s + \delta$$

if and only if $|x - s| < \delta$. But we also need that $x \neq s$. This condition is equivalent to requiring that $|x - s| > 0$. It follows that both conditions

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s$$

are satisfied simultaneously if and only if

$$0 < |x - s| < \delta.$$

We conclude that the function $f\colon \mathbb{R} \setminus \{s\} \to \mathbb{R}$ satisfies $\lim\limits_{x \to s} f(x) = L$ if and only if, given any positive real number ε, there exists a positive real number δ such that

$$|f(x) - L| < \varepsilon$$

for all real numbers x that satisfy

$$0 < |x - s| < \delta.$$

Example

Let s be a positive real number. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined such that

$$f(x) = \sqrt{1 + 3x^2}.$$

for all real numbers x. Now $f(1) = 2$. We investigate whether this function tends to the limit 2 as x tends to 1.

Now $\lim\limits_{x \to 1} f(x) = 2$ if and only if, given any positive real number ε, there exists some positive real number δ such that $|f(x) - 2| < \varepsilon$ for all real numbers x that satisfy $0 < |x - 1| < \delta$. Now $f(x)$ is an increasing function of x when $x > 0$. Taking account of this, we see that, given $\varepsilon > 0$, it suffices to discover a positive real number δ so that

$$f(1 + \delta) \leq 2 + \varepsilon$$

and

$$f(1 - \delta) \geq 2 - \varepsilon.$$

Indeed, because $f(x)$ increases with x when $x > 0$, if a value of δ is found that satisfies these inequalities then

$$2 - \varepsilon \leq f(1 - \delta) < f(x) < f(1 + \delta) \leq 2 + \varepsilon$$

Now

$$\begin{aligned} & f(1 + \delta) \leq 2 + \varepsilon \\ \iff \quad & \sqrt{1 + 3(1 + \delta)^2} \leq 2 + \varepsilon \\ \iff \quad & 1 + 3(1 + \delta)^2 \leq (2 + \varepsilon)^2 \\ \iff \quad & 4 + 6\delta + 3\delta^2 \leq 4 + 4\varepsilon + \varepsilon^2 \\ \iff \quad & 6\delta + 3\delta^2 \leq 4\varepsilon + \varepsilon^2. \end{aligned}$$

There is another inequality that needs to be satisfied, namely the inequality

$$f(1-\delta) \geq 2-\varepsilon.$$

We investigate what this inequality entails.

$$\begin{aligned} & f(1-\delta) \geq 2-\varepsilon \\ \iff & \sqrt{1+3(1-\delta)^2} \geq 2-\varepsilon \\ \iff & 1+3(1-\delta)^2 \geq (2-\varepsilon)^2 \\ \iff & 4-6\delta+3\delta^2 \geq 4-4\varepsilon+\varepsilon^2 \\ \iff & -6\delta+3\delta^2 \geq -4\varepsilon+\varepsilon^2 \\ \iff & 6\delta-3\delta^2 \leq 4\varepsilon-\varepsilon^2. \end{aligned}$$

It follows that the positive real number δ must be chosen so as to ensure that

$$6\delta+3\delta^2 \leq 4\varepsilon+\varepsilon^2 \quad \text{and} \quad 6\delta-3\delta^2 \leq 4\varepsilon-\varepsilon^2.$$

Having arrived at this stage, there is certainly more than one way to find a positive real number δ that satisfies these inequalities. We shall proceed by solving the quadratic equations that determine positive real numbers δ_1 and δ_2 for which

$$3\delta_1^2 + 6\delta_1 - 4\varepsilon - \varepsilon^2 = 0 \quad \text{and} \quad -3\delta_2^2 + 6\delta_2 - 4\varepsilon + \varepsilon^2 = 0.$$

The quadratic formula yields the results that

$$\begin{aligned} \delta_1 &= \frac{1}{3}(-3 + \sqrt{9 + 12\varepsilon + 3\varepsilon^2}) \\ \delta_2 &= \frac{1}{3}(3 - \sqrt{9 - 12\varepsilon + 3\varepsilon^2}) \end{aligned}$$

(The solutions resulting from application of the quadratic formula have been discarded, as δ must be close to zero when ε is close to zero. The discarded solutions correspond to values of δ_1 and δ_2 that bring $1 + \delta_1$ and $1 - \delta_2$ close to -1 for small ε). If we now take δ to be the minimum of δ_1 and δ_2 then $\delta > 0$ and $|f(x) - 2| < \varepsilon$ for all real numbers x that satisfy $0 < |x - 1| < \delta$.

4.8. Limit Points of Sets of Real Numbers

There is a technicality that needs to be addressed in order to formulate a definition of limit of a real-valued function defined over a subset D of the set $\mathbb{R}$ of real numbers that has the generality required in order to make use of the theory of limits to define concepts such as continuity and one-sided limits.

Suppose that we have a real-valued function $f \colon D \to \mathbb{R}$ defined over some subset D of the set of real numbers. For which real numbers s is it appropriate to consider whether or not the limit $\lim_{x \to s} f(x)$ exists?

Example

Let $f\colon [-1,1] \to \mathbb{R}$ be the function defined over the interval $[-1,1]$ so that $f(x) = \sqrt{1-x^2}$, where

$$[-1,1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}.$$

It would make no sense to consider what value, if any, could be the limit $\lim\limits_{x\to 100}$ of $f(x)$ as x tends to 100.

Example

Let $f\colon (-1,1) \to \mathbb{R}$ be the function defined over the interval $(-1,1)$ so that

$$f(x) = \frac{(x - x^2)\sqrt{1 - x^2}}{1 - x}.$$

It might well make sense to consider what value, if any, could be the limit $\lim_{x\ to 1}$ of $f(x)$ as x tends to 1, even though the number 1 lies outside the domain of the function. And indeed the function $f(x)$ tends to zero as x tends to 1.

The appropriate concept that determines those values of s at which limits can sensibly be taken is that of a *limit point*. This concept enters into the statement of formal propositions concerning limits, but its primary purpose is to ensure that limits, determined according to the formal definition of limits, are in fact uniquely determined according to that definition. Thus although it appears regularly in the statement of propositions, the defining property of limit points rarely appears in proofs.

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers. A real number s is a *limit point* of D if, given any positive real number δ, there exists a real number x belonging to D which satisfies $0 < |x - s| < \delta$.

To summarize: if $f\colon D \to \mathbb{R}$ is a real-valued function defined over a subset D of $\mathbb{R}$, and if s is a limit point of D, then it makes sense to consider whether or not the limit $\lim_{x\to s} f(x)$ has a well-defined value as x tends to s in D. If s is not a limit point of D then it makes no sense to consider whether or not this limit has a well-defined value.