MA1S11—Calculus Portion
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4.3. Limits of Polynomial Functions

Proposition 4.2

Let $p(x)$ be a polynomial and let s be a real number. Then

$$\lim_{x \to s} p(x) = p(s).$$

Proof

It follows from the Remainder Theorem (Theorem 2.6) that

$$p(x) = (x - s)q_s(x) + p(s),$$

where $q_s(x)$ is the polynomial obtained by dividing the polynomial $p(x)$ by the polynomial $x - s$, taking quotient $q_s(x)$ and remainder $p(s)$.

Let n be the degree of the polynomial p, and let

$$q_s(x) = b_0 + b_1 x + b_2 x^2 + \cdots b_{n-1} x^{n-1}$$

where the coefficients $b_0, b_1, \ldots, b_{n-1}$ are real numbers. Let R be a real number chosen so that $R \geq 1$ and $-R < s < R$, and let C be a positive real number chosen large enough to ensure that

$$-C \leq b_j \leq C$$

for $j = 0, 1, \ldots, n-1$. Then

$$-CR^n \leq -CR^j \leq b_j x^j \leq CR^j \leq CR^n$$

for $j = 0, 1, \ldots, n-1$ and for all real numbers x satisfying $-R \leq x \leq R$. It follows that

$$-nCR^n \leq q_s(x) \leq nCR^n$$

for all real numbers x satisfying $-R \leq x \leq R$.

Let some strictly positive real number ε be given. Then some positive real number δ can be chosen so as to ensure that

$$-R \leq s - \delta < s + \delta \leq R$$

and $nCR^n\delta \leq \varepsilon$. If a real real number x satisfies $s - \delta < x < s + \delta$ then $-nCR^n \leq q_s(x) \leq nCR^n$ and therefore

$$-\varepsilon \leq -nCR^n\delta < (x - s)q_s(x) < nCR^n\delta \leq \varepsilon.$$

But $(x - s)q_s(x) = p(x) - p(s)$. We have thus shown that

$$p(s) - \varepsilon < p(x) < p(s) + \varepsilon$$

for all real numbers x satisfying $s - \delta < x < s + \delta$. It follows that

$$\lim_{x \to s} p(x) = p(s),$$

as required. ∎

4.4. Sums of Geometric Sequences

We prove some well-known formulae concerning finite sums of geometric sequences.

Proposition 4.3

Let x be a real number, where $x \neq 1$. Then

$$\sum_{j=0}^{n-1} x^j = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

for all natural numbers n.

Proof

The identity

$$\sum_{j=0}^{n-1} x^j = \frac{x^n - 1}{x - 1}$$

is satisfied when $n = 1$. Indeed both sides of the identity have the value 1 when $n = 1$.

Let k be a natural number for which

$$\sum_{j=0}^{k-1} x^j = \frac{x^k - 1}{x - 1}.$$

Then

$$\begin{aligned}\sum_{j=0}^{k} x^j &= \sum_{j=0}^{k-1} x^j + x^k = \frac{x^k - 1}{x - 1} + x^k \\ &= \frac{x^k - 1}{x - 1} + \frac{x^k(x - 1)}{x - 1} = \frac{x^k - 1}{x - 1} + \frac{x^{k+1} - x^k}{x - 1} \\ &= \frac{x^{k+1} - 1}{x - 1}.\end{aligned}$$

Thus if the identity

$$\sum_{j=0}^{n-1} x^j = \frac{x^n - 1}{x - 1}$$

holds when $n = k$, where k is some natural number, then this identity holds when $n = k + 1$. It follows from the Principle of Mathematical Induction that this identity follows for all natural numbers n, as required. ∎

Corollary 4.4

Let n be a positive integer, and let u and v be distinct real numbers. Then

$$\begin{aligned}\frac{v^n - u^n}{v - u} &= u^{n-1} + u^{n-2}v + u^{n-2}v^2 + \cdots + uv^{n-2} + v^{n-1} \\ &= \sum_{j=0}^{n-1} u^{n-1-j}v^j.\end{aligned}$$

Proof

Let $x = \dfrac{v}{u}$. Then $x \neq 1$, because $u \neq v$, and $v^j = u^j x^j$ for all non-negative integers j. In particularly, $v - u = ux - u = u(x - 1)$ and $v^n - u^n = u^n(x^n - 1)$. It follows from Proposition 4.3 that

$$\frac{v^n - u^n}{v - u} = \frac{u^{n-1}(x^n - 1)}{x - 1} = u^{n-1}\left(\sum_{j=0}^{n-1} x^j\right) = \sum_{j=0}^{n-1} u^{n-1-j} v^j,$$

as required. ∎

4.5. Derivatives of Polynomial Functions

Proposition 4.5

Let n be a positive integer, and let s be a real number. Then

$$\lim_{x \to s} \frac{x^n - s^n}{x - s} = n s^{n-1}.$$

Proof

Let

$$q_s(x) = \sum_{j=0}^{n-1} s^{n-1-j} x^j.$$

for all real numbers x. Then $q_s(x)$ is a polynomial function of x. Moreover

$$\frac{x^n - s^n}{x - s} = q_s(x)$$

for all real numbers x distinct from s (see Corollary 4.4).

Now

$$\lim_{x \to s} q_s(x) = q_s(s) = ns^{n-1},$$

because $q_s(x)$ is a polynomial function of x (see Proposition 4.2). It follows that

$$\lim_{x \to s} \frac{x^n - s^n}{x - s} = ns^{n-1},$$

as required. ∎

The result and proof strategy of Proposition 4.5 can be generalized to obtain the derivative

$$\lim_{x \to s} \frac{p(x) - p(s)}{x - s}$$

of a polynomial function $p(x)$ at a particular value s of the real variable x.

Proposition 4.6

Let $p(x)$ be a polynomial function of x, and let

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n.$$

Let s be a real number. Then

$$\lim_{x \to s} \frac{p(x) - p(s)}{x - s} = a_1 + 2a_2 s + 3a_3 s^2 + \cdots + n a_n s^{n-1}$$

Proof

For each integer k between 1 and n, the kth power x^k of the real variable x satisfies the identity

$$\frac{x^k - s^k}{x - s} = q_{s,k}(x),$$

for all real numbers x distinct from x, where

$$q_{x,k}(x) = \sum_{j=0}^{k-1} s^{k-1-j} x^j$$

(see Corollary 4.4). Multiplying the identity satisfied by x^k by a_k and summing for $k = 1, 2, \ldots, n$, we find that

$$\frac{p(x) - p(s)}{x - s} = q_s(x),$$

where

$$q_s(x) = \sum_{k=1}^{n} a_k q_{s,k}(x).$$

Now $q_{s,k}(x)$ is a polynomial function of x for $k = 1, 2, \ldots, n$. It follows that $q_s(x)$ is a polynomial function of x, and therefore

$$\begin{aligned}\lim_{x \to s} q_s(x) &= q_s(s) = \sum_{k=1}^{n} a_k q_{s,k}(s) \\ &= \sum_{k=1}^{n} k a_k s^{k-1}.\end{aligned}$$

The result follows. ∎

Let $p(x)$ be a polynomial function of a real variable x. The *derivative* $p'(x)$ of the polynomial $p(x)$ is defined so that its value at a real number s satisfies

$$p'(s) = \lim_{x \to s} \frac{p(x) - p(s)}{x - s} = \lim_{h \to 0} \frac{p(s + h) - p(s)}{h}.$$

for all real numbers s. The derivative of $p(x)$ may also be denoted by the expressions

$$\frac{dp(x)}{dx} \qquad \frac{d}{dx}\,(p(x))\,.$$

Proposition 4.6 shows that if

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

then

$$p'(x) = a_1 + 2a_2 s + 3a_3 s^2 + \cdots + n a_n s^{n-1}$$

Thus, for example, if

$$p(x) = ax^3 + bx^2 + cx + d,$$

where a. b, c and d are real constants, then

$$p'(x) = 3ax^2 + 2bx + c.$$

4.6. Local Maxima and Minima of Polynomial Functions

Lemma 4.7

Let $p(x)$ be a polynomial, let s be a real number, and let $p'(s)$ be the derivative of p at s. Suppose that $p'(s) > 0$. Then there exists a positive real number δ such that $p(x) > p(s)$ for all real numbers x satisfying $s < x < s + \delta$ and $p(x) < p(s)$ for all real numbers x satisfying $s - \delta < x < s$.

Proof
It follows from the Remainder Theorem (Theorem 2.6) that a polynomial $q(x)$ can be determined so that

$$p(x) = (x - s)q(x) + p(s).$$

Now $q(s) = p'(s)$ (see Proposition 4.6). Moreover $q(s) = \lim_{x \to s} q(x)$ (see Proposition 4.2). Now $q(s) = p'(s) > 0$. It follows the definition of limits that there exists some strictly positive real number δ so that $q(x) > 0$ for all positive real numbers x satisfying $s - \delta < x < s + \delta$. But then the equation

$$p(x) = (x - s)q(x) + p(s).$$

ensures that $p(x) > p(s)$ for all real numbers x satisfying $s < x < s + \delta$, and $p(x) < p(s)$ for all real numbers x satisfying $s - \delta < x < s$. The result follows. $\blacksquare$

Proposition 4.8

Let $p(x)$ be a polynomial, let s be a real number, and let $p'(s)$ be the derivative of p at s. Suppose that the function $x \mapsto p(x)$ mapping each real number x to $p(x)$ has a local maximum or local minimum at $x = s$. Then $p'(s) = 0$.

Proof

It follows from Lemma 4.7 that if $p'(s) > 0$ then the function $x \mapsto p(x)$ cannot have a local maximum or local minimum at $x = s$. Applying this result with p replaced by $-p$, we see that if $p'(s) < 0$ then $(-p)'(s) > 0$, and therefore the function $x \mapsto -p(x)$ cannot have a local maximum or local minimum at $x = s$. It follows that if $p'(s) < 0$ then the function $x \mapsto p(x)$ itself cannot have a local maximum or local minimum at $x = s$. Thus if the function $x \mapsto p(x)$ *does* have a local maximum or local minimum at $x = s$, then the only remaining possibility is that $p'(s) = 0$. The result follows. ∎

Example

Let

$$p(x) = x^3 - 9x^2 + 24x - 16.$$

Then

$$\begin{aligned} p'(x) &= 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) \\ &= 3(x-2)(x-4). \end{aligned}$$

It follows that the local maxima and minima must be located at $x = 2$ and $x = 4$, and indeed $p(x)$ achieves a local minimum with value 0 at $x = 4$ and a local maximum with value 4 at $x = 2$.