MA1S11—Calculus Portion
School of Mathematics, Trinity College
Michaelmas Term 2016
Lecture 13 (October 25, 2016)

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Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined such that

$$f(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then the limit of $f(x)$ as x tends to 0 does not exist.

Indeed suppose that this limit were to exist, and suppose that it were equal to the real number L. Then (applying the "epsilon-delta" definition of limits with $\varepsilon = \frac{1}{4}$ there would exist some strictly positive real number δ with the property that

$$L - \tfrac{1}{4} < f(x) < L + \tfrac{1}{4}$$

for all non-zero real numbers x satisfying $-\delta < x < \delta$. It would then follow that

$$-\tfrac{1}{2} \leq f(u) - f(v) \leq \tfrac{1}{2}$$

for all non-zero real numbers u and v satisfying $-\delta < u < \delta$ and $-\delta < v < \delta$.

But, given such a real number δ let $u = -\frac{1}{2}\delta$ and $v = \frac{1}{2}\delta$. Then $f(u) = 0$ and $f(v) = 1$, and therefore $f(v) - f(u) = 1$. It follows that no strictly positive real number δ could exist with the stated properties, and thus the definition of limits cannot be satisfied by the function f at zero.

The example just discussed exemplifies the phenomenon that, for a function $f \colon D \to \mathbb{R}$ defined over a subset D of $\mathbb{R}$, the limit $\lim\limits_{x \to s} f(x)$ of $f(x)$ as x tends to some fixed value s will not exist if the function has a "jump" at s.

Example

Consider the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined such that

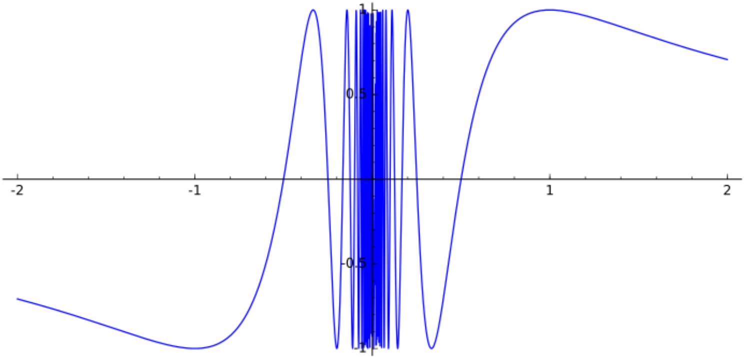
$$f(x) = \sin\left(\frac{\pi}{2x}\right)$$

for all non-zero real numbers x. First we review the values of $\sin(\frac{1}{2}\pi j)$ when j is an integer. These values are determined as follows:

$$\sin(\tfrac{1}{2}\pi j) = \begin{cases} 0 & \text{if } j \text{ is an even integer;} \\ 1 & \text{if } j - 1 \text{ is divisible by 4;} \\ -1 & \text{if } j - 3 \text{ is divisible by 4.} \end{cases}$$

It follows that

$$f(x) = \sin\left(\frac{\pi}{2x}\right) = \begin{cases} 0 & \text{if } x = \dfrac{1}{2k} \text{ for some non-zero integer } k; \\ 1 & \text{if } x = \dfrac{1}{4k+1} \text{ for some integer } k; \\ -1 & \text{if } x = \dfrac{1}{4k+3} \text{ for some integer } k. \end{cases}$$



Graph of $\sin\left(\frac{\pi}{2x}\right)$ as a function of x for $x \neq 0$.

(Plot generated using SageMath.)

The function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ does not tend to any limit as x tends to 0. Indeed suppose that this function were to converge to some limit L. Taking $\varepsilon = \frac{1}{4}$ in the “epsilon-delta” criterion that must be satisfied for a limit to exist, we see that there would exist some positive real number δ such that

$$L - \tfrac{1}{4} < f(x) < L + \tfrac{1}{4}$$

for all non-zero real numbers x satisfying $-\delta < x < \delta$. It would then follow that

$$-\tfrac{1}{2} < f(u) - f(v) < \tfrac{1}{2}$$

for all non-zero real numbers u and v satisfing $-\delta < u < \delta$ and $-\delta < v < \delta$. But a positive integer k could be chosen large enough to ensure that

$$\frac{1}{4k+1} < \delta.$$

Letting

$$u = \frac{1}{4k+1} \quad \text{and} \quad v = \frac{1}{4k+3},$$

it would be the case that $0 < u < \delta$, $0 < v < \delta$ and $f(u) - f(v) = 2$. But it would follow from the inequalities $0 < u < \delta$ and $0 < v < \delta$ that $f(u) - f(v) < \frac{1}{2}$, and thus a contradiction would arise were the limit of $f(x)$ as x tends to zero to exist. Therefore no such limit can exist.

Example

Consider the function $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined such that

$$g(x) = 3x \sin\left(\frac{\pi}{2x}\right)$$

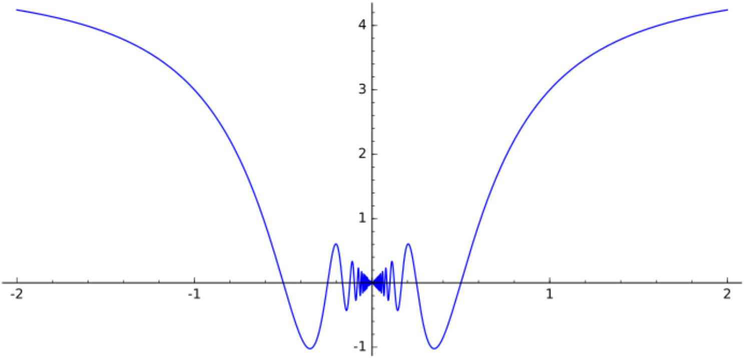
for all non-zero real numbers x. Now

$$-1 \leq \sin\left(\frac{\pi}{2x}\right) \leq 1$$

for all non-zero real numbers x. It follows that

$$-3|x| \leq g(x) \leq 3|x|$$

for all non-zero real numbers x.



Graph of $3x \sin\left(\frac{\pi}{2x}\right)$ as a function of x for $x \neq 0$.

(Plot generated using SageMath.)

Let some strictly positive real number ε be given. Let $\delta = \frac{1}{3}\varepsilon$. If a non-zero real number x satisfies

$$-\delta < x < \delta$$

then

$$-3\delta < g(x) < 3\delta,$$

and thus

$$-\varepsilon < g(x) < \varepsilon.$$

We conclude that

$$\lim_{x \to 0} g(x) = 0.$$

Example

Consider the function $h\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined such that

$$h(x) = 2\sqrt{|x|}\,\sin\left(\frac{\pi}{2x}\right)$$

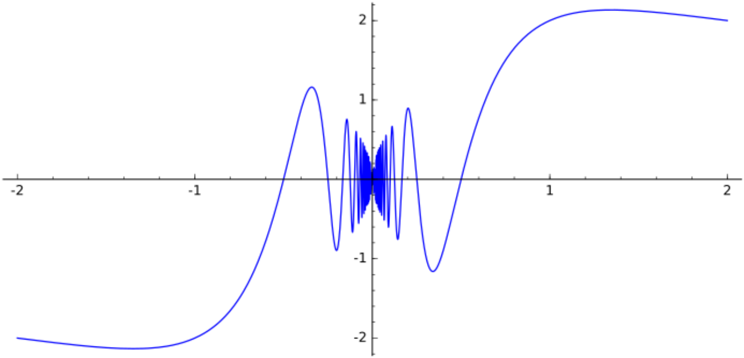
for all non-zero real numbers x. Now

$$-1 \leq \sin\left(\frac{\pi}{2x}\right) \leq 1$$

for all non-zero real numbers x. It follows that

$$-2\sqrt{|x|} \leq h(x) \leq 2\sqrt{|x|}$$

for all non-zero real numbers x.



Graph of $2\sqrt{|x|}\sin\left(\dfrac{\pi}{2x}\right)$ as a function of x for $x \neq 0$.

(Plot generated using SageMath.)

Let some strictly positive real number ε be given. Let $\delta = \frac{1}{4}\varepsilon^2$. If a non-zero real number x satisfies

$$-\delta < x < \delta$$

then

$$-2\sqrt{\delta} < h(x) < 2\sqrt{\delta},$$

and thus

$$-\varepsilon < h(x) < \varepsilon.$$

We conclude that

$$\lim_{x \to 0} h(x) = 0.$$