MA1S11—Calculus Portion
School of Mathematics, Trinity College
Michaelmas Term 2016
Lecture 10 (October 18, 2016)

David R. Wilkins

3.8. Increase and Decrease of Functions of a Real Variable

Definition

A function is said to be *real-valued* if its codomain is the set $\mathbb{R}$ of real numbers, or if its codomain is some subset of $\mathbb{R}$.

We consider the increase and decrease of real-valued functions whose domain is a subset of the set $\mathbb{R}$ of the real numbers, discussing intervals in the domain where such functions increase and decrease, and points in the domain at which such functions attain local minima and maxima.

Definition

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set $\mathbb{R}$ of real numbers. Then:

the function $f\colon D \to \mathbb{R}$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all elements u and v of D satisfying $u \leq v$;

the function $f\colon D \to \mathbb{R}$ is said to be (strictly) *increasing* if $f(u) < f(v)$ for all elements u and v of D satisfying $u < v$;

the function $f\colon D \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all elements u and v of D satisfying $u \leq v$;

the function $f\colon D \to \mathbb{R}$ is said to be (strictly) *decreasing* if $f(u) > f(v)$ for all elements u and v of D satisfying $u < v$.

Definition

A real-valued function $f\colon D \to \mathbb{R}$ defined over a subset D of the set $\mathbb{R}$ of real numbers is said to be *monotonic* if it is non-decreasing or non-increasing on D.

Lemma 3.4

Let $f\colon D \to \mathbb{R}$ be a (strictly) increasing function defined over a subset D of the set $\mathbb{R}$ of real numbers. Then the function $f\colon D \to \mathbb{R}$ is injective.

Proof

Let u and v be distinct elements of the set D. Then either $u < v$ or $v < u$. If $u < v$ then $f(u) < f(v)$, because the function f is increasing, and therefore $f(u) \neq f(v)$. If $v < u$ then $f(v) < f(u)$ and therefore $f(u) \neq f(v)$. The result follows. ■

Lemma 3.5

Let $f: D \to \mathbb{R}$ be a (strictly) decreasing function defined over a subset D of the set $\mathbb{R}$ of real numbers. Then the function $f: D \to \mathbb{R}$ is injective.

Proof

Let u and v be distinct elements of the set D. Then either $u < v$ or $v < u$. If $u < v$ then $f(u) > f(v)$, because the function f is decreasing, and therefore $f(u) \neq f(v)$. If $v < u$ then $f(v) > f(u)$ and therefore $f(u) \neq f(v)$. The result follows. ∎

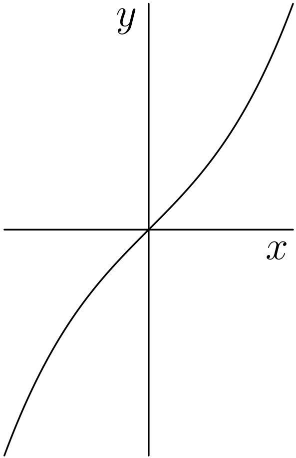
Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself defined such that $f(x) = x^3 + x$ for all real numbers x.

Let u and v be real numbers satisfying $u < v$. If $u \geq 0$ and $v \geq 0$ then $u^3 < v^3$. If $u < 0$ is negative and $v \geq 0$ is non-negative then $u^3 < 0$ and $v^3 \geq 0$ and therefore $u^3 < v^3$. If $u < 0$ and $v < 0$ then $u = -|u|$, $v = -|v|$. Moreover $|u| > |v|$, because $u < v$. It follows that

$$u^3 = (-|u|)^3 = -|u|^3 < -|v|^3 = (-|v)^3 = v^3.$$

The case when $v < 0$ and $u \geq 0$ does not arise, because $u < v$. We have therefore investigated all relevant cases determined by the signs of the real numbers u and v, and, in all cases, we have shown that $u^3 < v^3$. Thus $u^3 < v^3$ for all real numbers u and v satisfying $u < v$.



Graph of the curve $y = x^3 + x$.

Adding the inequalities $u^3 < v^3$ and $u < v$ we find that $u^3 + u < v^3 + v$ whenever $u < v$. Thus $f(u) < f(v)$ for all real numbers u and v satisfying $u < v$. It follows that the function $f: \mathbb{R} \to \mathbb{R}$ is increasing, where $f(x) = x^3 + x$ for all real numbers x. It follows from Lemma 3.4 that this function is injective.

Example

Let $g: \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself defined such that $g(x) = x^3 - x$ for all real numbers x. Then $g(-1) = g(0) = g(1)$. It follows that the function g is not injective.

Example

Let $a = \dfrac{1}{\sqrt{3}}$, let $h\colon [a, +\infty) \to \mathbb{R}$ be the function from $[a, +\infty)$ to $\mathbb{R}$ defined such that $h(x) = x^3 - x$ for all real numbers x satisfying $x \geq a$, and let $k\colon [-a, a] \to \mathbb{R}$ be the function defined such that $k(x) = x^3 - x$ for all real numbers x satisfying $-a \leq x \leq a$.

Let u and v be real numbers satisfying $a \leq u < v$. Then

$$v^3 - u^3 = (v - u)(v^2 + uv + u^2),$$

and therefore

$$h(v) - h(u) = (v - u)(v^2 + uv + u^2 - 1).$$

But $a \leq u < v$, and therefore

$$v^2 + uv + u^2 - 1 \geq 3a^2 - 1 = 0.$$

It follows that $h(v) > h(u)$ for all real numbers u and v satisfying $a \leq u < v$. Thus the function $h\colon [a, +\infty) \to \mathbb{R}$ is increasing on $[a, +\infty)$, where $a = \frac{1}{\sqrt{3}}$, and therefore this function h is injective.

Next let u and v be real numbers satisfying $-a \leq u < v \leq a$. Then

$$k(v) - k(u) = (v - u)(v^2 + uv + u^2 - 1).$$

If $u \geq 0$ and $v \geq 0$ then

$$v^2 + uv + u^2 - 1 < 3a^2 - 1 = 0$$

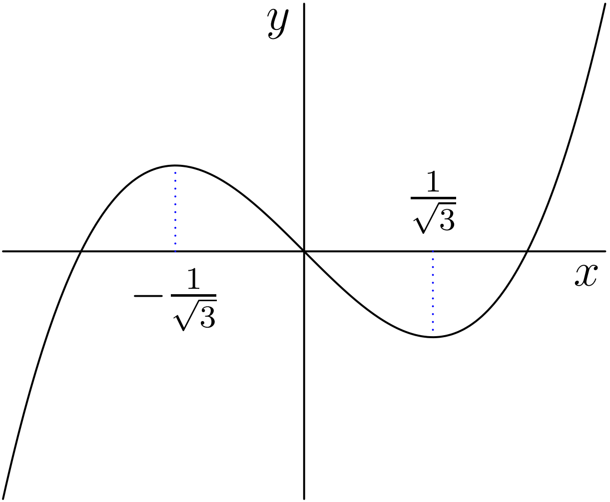
If $u < 0$ and $v < 0$ then

$$v^2+uv+u^2-1 = (-|v|)^2+(-|u|)(-|v|)+(-|u|)^2-1 < 3a^2-1 = 0,$$

and if $u < 0$ and $v \geq 0$ then $v^2 + uv + u^2 - 1 \leq -1$. It follows that

$$k(v) - k(u) = (v - u)(v^2 + uv + u^2 - 1) < 0$$

for all real numbers u and v satisfying $-a \leq u < v \leq a$. Thus the function $k\colon [-a, a] \to \mathbb{R}$ is decreasing on $[-a, a]$, where $a = \frac{1}{\sqrt{3}}$, and therefore this function k is injective.



Graph of the curve $y = x^3 - x$.

We now resume discussion of the function $g\colon \mathbb{R} \to \mathbb{R}$, where $g(x) = x^3 - x$ for all real numbers x. Let $a = \dfrac{1}{\sqrt{3}}$. We have proved that the function h obtained by restricting the function g to the interval $[a, +\infty)$ is an increasing function. It follows that $g(x) > g(a)$ whenever $x > a$. We have also proved that the function k obtained by restricting the function g to the interval $[-a, a]$ is a decreasing function. It follows that $g(x) > g(a)$ whenever $-a \leq x < a$. We conclude that $g(x) \geq g(a)$ for all real numbers x satisfying $x \geq -a$. This justifies the assertion that $g(x)$ attains a *local minimum* when $x = \frac{1}{\sqrt{3}}$.

We now obtain the corresponding result that $g(x)$ attains a *local maximum* when $x = -a$. Now the function $g\colon \mathbb{R} \to \mathbb{R}$ is an *odd* function. This means that $g(-x) = -g(x)$ for all real numbers x. It follows that $g(x) = -h(|x|)$ for all real numbers x satisfying $x < -a$, where $h\colon [a, +\infty) \to \mathbb{R}$ is defined such that $h(x) = x^3 - x$ for all real numbers x satisfying $x \geq a$. Now we have shown that the function h is an increasing function on the interval $[a, +\infty)$. It follows that if u and v are real numbers satisfying $u < v \leq -a$ then $a \leq |v| < |u|$. But then $h(|u|) > h(|v|)$, because the function h is increasing on $[a, +\infty)$. It follows that

$$g(u) = -h(|u|) < -h(|v|) = g(v).$$

We conclude from this that the function g is increasing on the interval $(-\infty, -a]$, where $a = \frac{1}{\sqrt{3}}$. We have already shown that the function g is decreasing on the interval $[-a, a]$ (because it is equal on this interval to the function $k\colon [-a, a] \to \mathbb{R}$, and we have shown that the function k is decreasing). It follows that $g(x) < g(-a)$ when $x < -a$ and $g(x) < g(-a)$ when $-a < x < a$. We conclude from this that $g(x) \leq g(-a)$ for all real numbers x satisfying $x \leq a$. This justifies the assertion that $g(x)$ attains a *local maximum* when $x = -\frac{1}{\sqrt{3}}$.

Definition

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set $\mathbb{R}$ of real numbers. We say that the function f attains a *local minimum* at an element s of D if there exists some positive real number δ such that $f(x) \geq f(s)$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$.

Definition

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set $\mathbb{R}$ of real numbers. We say that the function f attains a *local maximum* at an element s of D if there exists some positive real number δ such that $f(x) \leq f(s)$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$.

We now introduce the concept of a *neighbourhood* of a real number s in some subset D of $\mathbb{R}$ to which the real number s belongs.

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, and let the real number s be an element of D. A subset N of D is said to be a *neighbourhood* of s (in D) if there exists some positive real number δ such that $x \in N$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$.

The formal definition of *neighbourhood* captures the notion that a subset N of D is a neighbourhood of the real number s if and only if N contains all elements of D that lie “sufficiently close” to s.

Another perhaps useful way of thinking about neighbourhoods is to observe that a subset N of a subset D of the set of real numbers is a neighbourhood of some element s of D if and only if N "completely surrounds" s in D, so that s cannot be "approached" within D without entering into the neighbourhood of s. (In the same way, one cannot approach the house of a friend without passing through a neighbourhood within which the house is located.)

The definitions of local maxima and minima may now be reformulated in perhaps more attractive terms as presented in the following lemmas, which follow directly from the relevant definitions.

Lemma 3.6

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set $\mathbb{R}$ of real numbers. The function f attains a local minimum at an element s of D if and only if there exists some neighbourhood N of s in D small enough to ensure that $f(x) \geq f(s)$ for all $x \in N$.

Lemma 3.7

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set $\mathbb{R}$ of real numbers. The function f attains a local maximum at an element s of D if and only if there exists some neighbourhood N of s in D small enough to ensure that $f(x) \leq f(s)$ for all $x \in N$.

Example

Let a and c be real constants, where $a > 0$ and $c > 0$, and let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function defined on the set $\mathbb{R} \setminus \{0\}$ of non-zero real numbers so that

$$f(x) = ax + \frac{c}{x}$$

for all non-zero real numbers x. We shall investigate the qualitative behaviour of this function, and will in particular determine the range of the function f.

Let y be a real number belonging to the range of the function f, and let x be a non-zero real number satisfying $f(x) = y$. Then

$$y = ax + \frac{c}{x}.$$

If we multiply both sides of this identity by x, and then subtract the left hand side from the right hand side, we arrive at the equation

$$ax^2 - yx + c = 0.$$

The quadratic polynomial on the left hand side of this equation must have real roots if y is to belong to the range of the function f. It follows from the standard quadratic formula that y must satisfy the inequality $y^2 \geq 4ac$.

Conversely if the real number y satisfies the inequality $y^2 \geq 4ac$ then the polynomial has real roots, and y therefore belongs to the range of the function f. In particular, if $y^2 = 4ac$ then there is a unique non-zero real number x for which $f(x) = y$. Moreover the unique real number x_0 for which $f(x_0) = 2\sqrt{ac}$, is given by the formula

$$x_0 = \frac{y}{2a} = \frac{2\sqrt{ac}}{2a} = \sqrt{\frac{c}{a}},$$

and the unique real number x for which $f(x) = -2\sqrt{ac}$, is

$$x = \frac{y}{2a} = -\frac{2\sqrt{ac}}{2a} = -\sqrt{\frac{c}{a}} = -x_0.$$

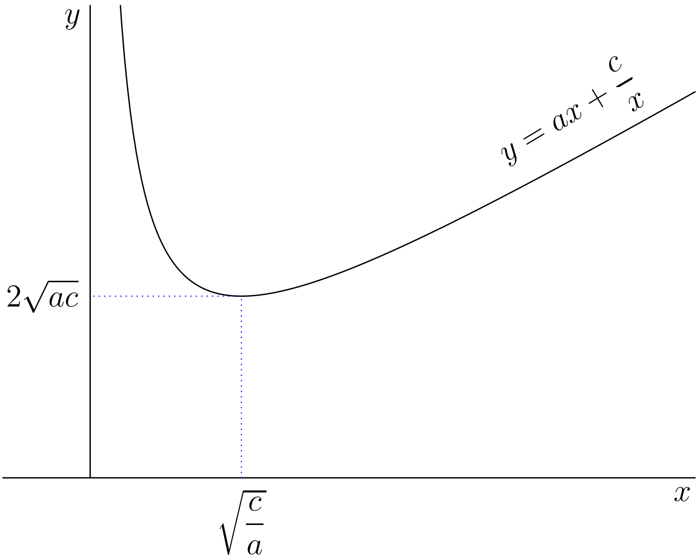
Next suppose that either $y > 2\sqrt{ac}$ or $y < -2\sqrt{ac}$. Then there exist two distinct non-zero real numbers x satisfying $f(x) = y$. They are

$$x = \frac{y \pm \sqrt{y^2 - 4ac}}{2a}.$$

We see from this that the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is not injective. The range of this function is the union

$$(-\infty, -2\sqrt{ac}] \cup [2\sqrt{ac}, +\infty).$$

It is now clear that the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is is not surjective.



y
$y = ax + \frac{c}{x}$
$2\sqrt{ac}$
x
$\sqrt{\frac{c}{a}}$

Now it is clear that there exists a positive real number δ_0 such that $f(x) \geq f(x_0) = 2\sqrt{ac}$ whenever $x_0 - \delta_0 < x < x_0 + \delta_0$, where

$$x_0 = \sqrt{\frac{c}{a}}.$$

Indeed it suffices to ensure that all non-zero real numbers satisfying these inequalities are positive, and thus we may pick $\delta_0 = x_0$, or alternatively we may set δ_0 equal to any positive real number not exceeding x_0. In these circumstances we say that $f(x) \geq f(x_0)$ throughout some *neighbourhood* of x_0, and accordingly we say that the function f has a local minimum at x_0, where $x_0 = \sqrt{c/a}$.

Similarly we say that the function f has a *local maximum* at $-x_0$, where $x_0 = \sqrt{c/a}$.

We now show formally that the function f is increasing on the interval $[x_0, +\infty)$, where $x_0 = \sqrt{c/a}$. Let u and v be real numbers satisfying $x_0 \leq u < v$. Then

$$\begin{aligned} f(v) - f(u) &= av + \frac{c}{v} - au - \frac{c}{u} \\ &= \left(a - \frac{c}{uv}\right)(v - u). \end{aligned}$$

Now
$$v > u \geq x_0 = \sqrt{\frac{c}{a}},$$
and therefore
$$\frac{c}{uv} < \frac{c}{u^2} \leq \frac{c}{x_0^2} = a.$$
It follows that
$$\frac{f(v) - f(u)}{v - u} = a - \frac{c}{uv} > 0$$
whenever $x_0 \leq u < v$, and therefore the function f is increasing on the interval $[x_0, +\infty)$.

Moreover, interchanging the roles of u and v, we find that the identity

$$\frac{f(v) - f(u)}{v - u} = a - \frac{c}{uv}$$

is valid for all real numbers u and v satisfying $u \geq x_0$, $v \geq x_0$ and $u \neq v$, irrespective of whether $u < v$ or $v < u$.

We now consider the behaviour of the function f on the interval $(0, x_0]$, where $x_0 = \sqrt{c/a}$. Let u and v be real numbers satisfying $u < v \leq x_0$. Then

$$f(v) - f(u) = \left(a - \frac{c}{uv}\right)(v - u).$$

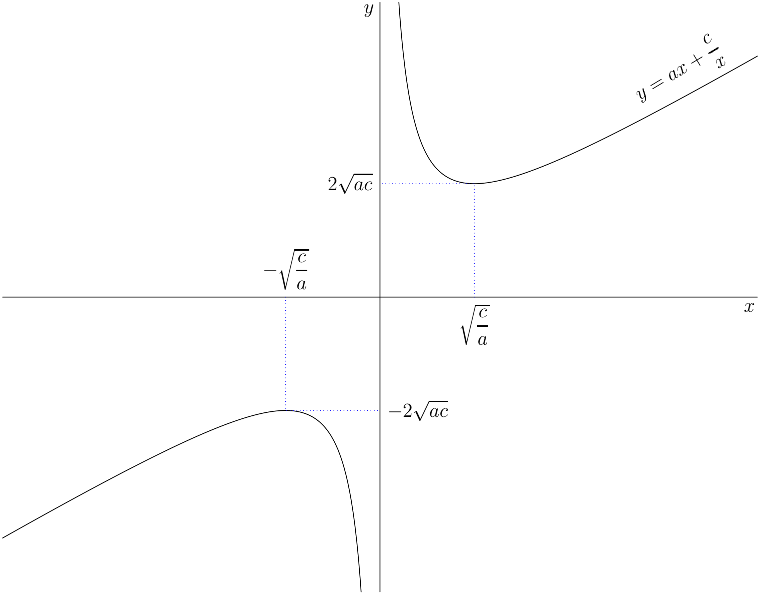
But now

$$\frac{c}{uv} > \frac{c}{v^2} \geq \frac{c}{x_0^2} = c \times \frac{a}{c} = a,$$

and therefore

$$a - \frac{c}{uv} < 0.$$

it follows that $f(v) < f(u)$ for all real numbers u and v satisfying $u < v < x_0$, and thus the function f is decreasing on the interval $(0, x_0]$, where $x_0 = \sqrt{c/a}$.



y
$y = ax + \frac{c}{x}$
$2\sqrt{ac}$
$-\sqrt{\frac{c}{a}}$
x
$\sqrt{\frac{c}{a}}$
$-2\sqrt{ac}$

The behaviour of $f(x)$ when $x < 0$ may be determined from the results already obtained in view of the fact that the function f is an *odd function* which satisfies the identity $f(-x) = -f(x)$ for all non-zero real numbers x. If u and v are real numbers satisfying $u < v \leq -x_0 < 0$, where $x_0 = \sqrt{c/a}$, then $|u| > |v| \geq x_0$, and therefore

$$f(v) - f(u) = -f(|v|) + f(|u|) > 0.$$

Thus the function f is increasing on the interval $(-\infty, -x_0]$. Similarly if u and v are real numbers satisfying $-x_0 \leq u < v < 0$ then $0 \leq |v| < |u| \leq x_0$, and therefore

$$f(v) - f(u) = -f(|v|) + f(|u|) < 0.$$

Thus the function f is decreasing on the interval $[-x_0, 0)$, where $x_0 = \sqrt{c/a}$.