MA1S11—Calculus Portion
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3. Functions

3.1. Functions between Sets

Definition

Let X and Y be sets. A *function* $f\colon X \to Y$ from X to Y assigns to each element x of the set X a corresponding element $f(x)$ of the set Y. The set X on which the function is defined is referred to as the *domain* of the function $f\colon X \to Y$. The set Y that contains the values of the function is referred to as the *codomain* of the function.

The following observations are important.

- The is no restriction on the nature of the contents of the sets X and Y appearing as the domain or codomain of a function. These sets could for example contain numbers, or words, or strings of characters representing DNA sequences, or colours, or students registered for a particular module.
- A function $f\colon X \to Y$ with domain X and codomain Y must assign a value $f(x)$ in Y to every single element x of the set X. Otherwise the definition of a function with domain X and codomain Y is not satisfied.

- In order to specify a function completely, it is necessary to specify both the domain X and the codomain Y of the function $f\colon X \to Y$.
- Algebraic formulae such as $\dfrac{x + 2x^3 + 3x^4}{\sqrt{1 - x^2}}$ may play a significant role in the specification of functions, but the concept of an algebraic expression of this sort is distinct from the concept of a function. Moreover a single algebraic expression of this sort is often not sufficient to specify some given mathematical function.

Example

There is no function with domain $\mathbb{R}$ and codomain $\mathbb{R}$ that maps x to $1/x$ for all real numbers x. This is because $1/x$ is not a well-defined real number when $x = 0$. The reciprocal function is regarded as a function

$$r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

defined so that

$$r(x) = \frac{1}{x} \quad \text{for all non-zero real numbers } x.$$

The set $\mathbb{R} \setminus \{0\}$ may be regarded as the "natural domain" of the reciprocal function. It is obtained from the set $\mathbb{R}$ of real numbers by subtracting from it the singleton set $\{0\}$ consisting of just the real number 0. It follows that this set $\mathbb{R} \setminus \{0\}$ is the set consisting of all non-zero real numbers.

Example

There is a well-defined function $f\colon [-1,1] \to \mathbb{R}$ defined so that

$$f(x) = \sqrt{1 - x^2} \quad \text{for all } x \in [-1,1],$$

where, in accordance with standard notation for intervals,

$$[-1,1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}.$$

It should be noted that, in accordance with a standard convention in mathematics, and in cases where u is a non-negative real number, the symbol $\sqrt{u}$ denotes the unique *non-negative* real number satisfying $(\sqrt{u})^2 = u$.

Note that if $g\colon D \to \mathbb{R}$ is a function with domain D and codomain $\mathbb{R}$, where D is a subset of the set $\mathbb{R}$ of real numbers, and if the function g is defined so that $g(x) = \sqrt{1 - x^2}$ for all $x \in D$, then it must be the case that $D \subset [-1, 1]$. Thus $[-1, 1]$ is the largest subset of the set $\mathbb{R}$ of the real numbers that can serve as the domain of a function mapping each real number x belonging to that domain to a corresponding real number $\sqrt{1 - x^2}$. For this reason one may regard $[-1, 1]$ as being the "natural domain" of a function f, mapping into the set of real numbers, that has the property that $f(x) = \sqrt{1 - x^2}$ for all elements x of the domain of the function.

3.2. Injective Functions

Let X be a set, and let u and v be elements of X. We say that u and v are *distinct* if $u \neq v$.

Definition

A function $f: X \to Y$ from a set X to a set Y is *injective* if it maps distinct elements of the set X to distinct elements of the set Y.

Let X and Y be sets, and let $f: X \to Y$ be a function from X to Y. Then $f: X \to Y$ is injective if and only if $f(u) \neq f(v)$ for all elements u and v of the domain X of the function that satisfy $u \neq v$.

One can show that a function $f\colon X \to Y$ is injective by proving that

$$u, v \in X \text{ and } f(u) = f(v) \implies u = v.$$

(In other words, one can show that a function $f\colon X \to Y$ is injective by proving that, if u and v are elements of the domain of the function, and if $f(u) = f(v)$, then $u = v$.)

Example

Let $f\colon [0, +\infty) \to \mathbb{R}$ be the function defined such that

$$f(x) = \frac{x}{x+1}$$

for all non-negative real numbers x. Let u and v be non-negative real numbers. Suppose that $f(u) = f(v)$. Then

$$\frac{u}{u+1} = \frac{v}{v+1}.$$

Multiplying both sides of this equation by $(u+1)(v+1)$, we find that

$$u(v+1) = v(u+1).$$

and thus

$$uv + u = uv + v.$$

Subtracting uv from both sides of this equation, we find that $u = v$. This proves that the function $f\colon [0, +\infty) \to \mathbb{R}$ is injective.

Lemma 3.1

Let X and Y be sets, and let $f: X \to Y$ be a function from X to Y. The function f is injective if and only if, given any element y of the codomain Y, there exists at most one element x of the domain X satisfying $f(x) = y$.

Proof

The result follows directly from the definitions. Indeed if the function is injective, then there cannot exist distinct elements of X that map to the same element of Y, and thus each element of Y is the image of at most one element of X. Conversely if each element of Y is the image of at most one element of X then distinct elements of X cannot map to the same element of Y, and therefore distinct elements of X must map to distinct elements of Y, and thus the function $f: X \to Y$ is injective.

3.3. The Range of a Function

Definition

Let X and Y be sets, and let $f\colon X \to Y$ be a function from X to Y. The *range* of $f\colon X \to Y$, often denoted by $f(X)$, consists of those elements of the codomain Y that are of the form $f(x)$ for at least one element x of X.

Let $f: X \to Y$ be a function from a set X to a set Y, and let $f(X)$ be the range of this function. Then

$$\begin{aligned} f(X) &= \{y \in Y \mid y = f(x) \text{ for some } x \in X\} \\ &= \{f(x) \mid x \in X\}. \end{aligned}$$

(We use above a standard notation from set theory: the specification $\{f(x) \mid x \in X\}$ denotes the set consisting of all objects that are of the form $f(x)$ for some element x of the set X.)

3.4. Surjective Functions

Definition

Let X and Y be sets. A function $f\colon X \to Y$ is said to be *surjective* if, given any element y of the codomain Y, there exists at least one element x of the domain X that satisfies $f(x) = y$.

The following result follows directly from the relevant definitions.

Lemma 3.2

A function $f\colon X \to Y$ from a set X to a set Y is surjective if and only if $f(X) = Y$, where $f(X)$ is the range of the function f, and Y is the codomain of the function.

Example

Consider the function $f\colon \mathbb{Z} \to \mathbb{Z}$ from the set $\mathbb{Z}$ of integers to itself defined such that $f(x) = 5x - 3$ for all $x \in \mathbb{Z}$. If u and v are integers, and if $f(u) = f(v)$ then $5u - 3 = 5v - 3$. Adding 3 to both sides of this equation, and then dividing by 5, we find that $5u = 5v$, and therefore $u = v$. We conclude that the function $f\colon \mathbb{Z} \to \mathbb{Z}$ is injective. Thus every integer k satisfies $f(n) = k$ for at most one integer n.

Now an integer k is of the form $f(n)$ for some integer n if and only if $k + 3$ is divisible by 5. It follows that the range $f(\mathbb{Z})$ of the function f consists of those positive integers whose decimal representation has least significant digit 2 or 7, together with those negative integers whose decimal representation has least significant digit 8 or 3. The range of the function is not equal to the codomain of the function, and therefore the function is not surjective.

Example

Consider the function $p\colon \mathbb{Z} \to \mathbb{Z}$ defined such that

$$p(n) = \begin{cases} n+3 & \text{if } n \text{ is an odd integer;} \\ n-5 & \text{if } n \text{ is an even integer.} \end{cases}$$

First we note that $p(n)$ is odd for all even integers n, and $p(n)$ is even for all odd integers n. We determine whether or not the function p is injective, and whether or not it is surjective.

We consider whether $p\colon \mathbb{Z} \to \mathbb{Z}$ is injective, where

$$p(n) = \begin{cases} n+3 & \text{if } n \text{ is an odd integer;} \\ n-5 & \text{if } n \text{ is an even integer.} \end{cases}$$

Let m and n be integers for which $p(m) = p(n)$. Suppose first that $p(m)$ is even. Then m and n are odd, and

$$m = p(m) - 3 = p(n) - 3 = n.$$

Next suppose that $p(m)$ is odd. Then m and n are even, and

$$m = p(m) + 5 = p(n) + 5 = n.$$

Now $p(m)$ is either even or odd. It follows that $m = n$ whenever $p(m) = p(n)$. Thus the function p is injective.

We consider whether $p\colon \mathbb{Z} \to \mathbb{Z}$ is surjective, where

$$p(n) = \begin{cases} n + 3 & \text{if } n \text{ is an odd integer;} \\ n - 5 & \text{if } n \text{ is an even integer.} \end{cases}$$

Let k be an integer. If k is even then $k = p(k - 3)$. If k is odd then $k = p(k + 5)$. Every integer k is even or odd. It follows that every integer is in the range of the function p, and thus this function is surjective.

We have thus shown that the function $p\colon \mathbb{Z} \to \mathbb{Z}$ is both injective and surjective.

3.5. Bijective Functions

Definition

A function $f: X \to Y$ from a set X to a set Y is said to be *bijective* if it is both injective and surjective.

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself. defined such that $f(x) = 5x - 3$ for all real numbers x. The function f is injective. Indeed if u and v are real numbers, and if $f(u) = f(v)$ then $5u - 3 = 5v - 3$. But then $5u = 5v$, and therefore $u = v$.

Now $y = f(\frac{1}{5}(y + 3))$ for all real numbers y. It follows that every element of the codomain $\mathbb{R}$ of the function is in the range of the function. This demonstrates that the function $f\colon X \to Y$ is surjective (see Lemma 3.2).

The function $f\colon \mathbb{R} \to \mathbb{R}$ has been shown to be both injective and surjective. It is therefore bijective.

3.6. Inverses of Functions

Definition

Let X and Y be sets, and let $f\colon X \to Y$ be a function from X to Y. A function $g\colon Y \to X$ is said to be the *inverse* of $f\colon X \to Y$ if $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$.

Example

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be the functions defined such that $f(x) = 5x - 3$ for all real numbers x and $g(y) = \frac{1}{5}(y + 3)$ for all real numbers y. Then

$$g(f(x)) = \frac{1}{5}(f(x) + 3) = \frac{1}{5}(5x) = x$$

for all real numbers x, and

$$f(g(y)) = 5 \times \frac{1}{5}(y + 3) - 3 = (y + 3) - 3 = y$$

for all real numbers y. It follows that the function $g: \mathbb{R} \to \mathbb{R}$ is the inverse of the function $f: \mathbb{R} \to \mathbb{R}$.

Example

Let $p\colon \mathbb{Z} \to \mathbb{Z}$ be the function from the set $\mathbb{Z}$ of integers to itself defined such that

$$p(n) = \begin{cases} n + 3 & \text{if } n \text{ is an odd integer;} \\ n - 5 & \text{if } n \text{ is an even integer.} \end{cases}$$

We have already shown in an earlier example that this function is both injective and surjective. It is therefore bijective.

Let $q\colon \mathbb{Z} \to \mathbb{Z}$ be the function from the set $\mathbb{Z}$ of integers to itself defined such that

$$q(m) = \begin{cases} m + 5 & \text{if } m \text{ is an odd integer;} \\ m - 3 & \text{if } m \text{ is an even integer.} \end{cases}$$

If n is an odd integer then $p(n)$ is an even integer, and $p(n) = n + 3$, and therefore $q(p(n)) = p(n) - 3 = n$. If n is an even integer then $p(n)$ is an odd integer and $p(n) = n - 5$, and therefore $q(p(n)) = p(n) + 5 = n$. It follows that $p(n) = n$ for all integers n, irrespective of whether n is even or odd.
A similar argument shows that $p(q(m)) = m$ for all integers m. It follows that the function $q\colon \mathbb{Z} \to \mathbb{Z}$ is the inverse of the function $p\colon \mathbb{Z} \to \mathbb{Z}$.

Proposition 3.3

Let X and Y be sets. A function $f: X \to Y$ from X to Y has a well-defined inverse $g: Y \to X$ if and only if $f: X \to Y$ is bijective.

Proof

First suppose that $f: X \to Y$ has a well-defined inverse $g: X \to Y$. We show that the function $f: X \to Y$ is then both injective and surjective.

Let u and v be elements of the domain X of f. Suppose that $f(u) = f(v)$. Then

$$u = g(f(u)) = g(f(v)) = v.$$

It follows that the function $f: X \to Y$ is injective. Also $y = f(g(y))$ for all elements y of the codomain Y of f, and therefore the function $f: X \to Y$ is surjective. We have thus proved that if $f: X \to Y$ has a well-defined inverse, then this function is bijective.

Now let $f: X \to Y$ be a function that is bijective. We must show that this function has a well-defined inverse $g: Y \to X$. Now the function $f: X \to Y$ is surjective, and therefore, given any element y of the codomain Y of f, there exists at least one element x of the domain X of f satisfying $f(x) = y$, because the function f is surjective. Also the function $f: X \to Y$ is injective, and therefore, given any element y of the codomain Y of f, there exists at most one element x of the domain X of f satisfying $f(x) = y$ (see Lemma 3.1). Putting these results together, we see that, given any element y of the codomain Y of f, there exists exactly one element x of the domain X of f satisfying $f(x) = y$. There therefore exists a function $g: Y \to X$ defined such that, given any element y of Y, $g(y)$ is the unique element of the set X that satisfies $f(g(y)) = y$.

Now the very definition of the function g ensures that $f(g(y)) = y$ for all $y \in Y$. We must also show that $g(f(x)) = x$ for all $x \in X$. Now, given $x \in X$, the element $g(f(x))$ is by definition the unique element u of the set X satisfying $f(u) = f(x)$. But the injectivity of the function f ensures that if $u \in X$ satisfies $f(u) = f(x)$ then $u = x$. It follows that $g(f(x)) = x$ for all elements x of the domain X of f. We have now verified that the bijective function $f\colon X \to Y$ does indeed have a well-defined inverse $g\colon Y \to X$. This completes the proof. $\blacksquare$

3.7. Natural Domains

Example

We consider what is the natural domain of a real-valued function f, where

$$f(x) = \sqrt{\frac{8}{\sqrt{x-1}+1} - 2}$$

for all elements x of this natural domain. (For the purposes of this example, we adopt the requirement that the square root $\sqrt{u}$ of a number is defined only when that number u is both real and non-negative.)

Now the inner square root needs to be defined. We therefore require that $x \geq 1$ for all elements x of the sought natural domain. Note that $\sqrt{x-1}+1 \geq 1$ whenever $x \geq 1$, and therefore

$$\frac{8}{\sqrt{x-1}+1}$$

is defined for all real numbers x satisfying $x \geq 1$.

But it is also necessary to ensure that

$$\frac{8}{\sqrt{x-1}+1} - 2 \geq 0$$

for all elements x of the natural domain of the function. We therefore require that

$$\frac{8}{\sqrt{x-1}+1} \geq 2.$$

Now

$$\begin{aligned} & \frac{8}{\sqrt{x-1}+1} \geq 2. \\ \iff \quad & \sqrt{x-1}+1 \leq 4 \\ \iff \quad & \sqrt{x-1} \leq 3 \\ \iff \quad & 0 \leq x-1 \leq 9 \\ \iff \quad & 1 \leq x \leq 10 \end{aligned}$$

(Here the symbol $\iff$ means "if and only if".)

It follows that the expression specifying the value of $f(x)$ is well-defined only if $1 \leq x \leq 10$. Moreover this expression does yield a well-defined real number for all real numbers x satisfying $1 \leq x \leq 10$. The natural domain of a function determined by the given expression is thus $[1, 10]$, where

$$[1, 10] = \{x \in \mathbb{R} \mid 1 \leq x \leq 10\}.$$

Thus the real-valued function with the most extensive domain specified by the given expression is the function

$$f\colon [1, 10] \to \mathbb{R}$$

defined such that

$$f(x) = \sqrt{\frac{8}{\sqrt{x - 1} + 1} - 2}$$

for all $x \in [1, 10]$. One can say that $[1, 10]$ is the "natural domain" for a function determined by the given expression.