MA1S11—Calculus Portion
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2.4. Polynomial Division

Example

Let $p(x)$ be the polynomial in x defined so that

$$p(x) = x^3 - 8x^2 + 17x - 10.$$

Now $p(1) = 0$. (Indeed the coefficients 1, -8, 17 and -10 add up to zero.) The problem is to find the other roots.

A standard procedure for discovering the other roots is to divide the polynomial $p(x)$ by the polynomial $x - 1$ using a calculation scheme modelled on a standard scheme for performing long division in arithmetic. The calculation goes as follows:—

2. Polynomials (continued)

$$
\begin{array}{r|rrrr}
 & x^2 & -7x & +10 & \\
\hline
x-1 & x^3 & -8x^2 & +17x & -10 \\
 & x^3 & -x^2 & & \\
\cline{2-3}
 & & -7x^2 & +17x & \\
 & & -7x^2 & +7x & \\
\cline{3-4}
 & & & 10x & -10 \\
 & & & 10x & -10 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

This calculation yields the result that

$$x^3 - 8x^2 + 17x - 10 = (x-1)(x^2 - 7x + 10).$$

Now the polynomial $x^2 - 7x + 10$ can be factored using the standard formula for the roots of a quadratic polynomial. Alternatively, because the leading term is equal to one, it follows from Lemma 2.2 that the sum of the roots of the polynomial $x^2 - 7x + 10$ is equal to 7 and the product of those roots is equal to 10. From this we can deduce that the roots are 2 and 5, and thus

$$x^2 - 7x + 10 = (x - 2)(x - 5),$$

and thus

$$p(x) = x^3 - 8x^2 + 17x - 10 = (x - 1)(x - 2)(x - 5).$$

We now divide the polynomial $x^3 - 8x^2 + 17x - 10$ by $x - 1$ using standard algebraic notation, to see how the individual steps are justified.

Let

$$p(x) = x^3 - 8x^2 + 17x - 10.$$

First we note that we can obtain a polynomial whose leading term matches the leading term x^3 of $p(x)$ by multiplying the polynomial $x - 1$ by x^2. Now $x^3 = (x - 1)x^2 + x^2$. It follows that

$$\begin{aligned} p(x) &= (x-1)x^2 + x^2 - 8x^2 + 17x - 10 \\ &= (x-1)x^2 - 7x^2 + 17x - 10. \end{aligned}$$

Next we note that we can obtain a polynomial whose leading term is $-7x^2$ by multiplying the polynomial $x - 1$ by $-7x$. Now $-7x^2 = -7(x - 1)x - 7x$. It follows that

$$\begin{aligned} p(x) &= x^3 - 8x^2 + 17x - 10 \\ &= (x-1)x^2 - 7x^2 + 17x - 10 \\ &= (x-1)x^2 - 7(x-1)x - 7x + 17x - 10 \\ &= (x-1)(x^2 - 7x) + 10x - 10. \end{aligned}$$

But $10x - 10 = 10(x-1)$. It follows that

$$p(x) = (x-1)(x^2 - 7x + 10).$$

Moreover $x^2 - 7x + 10 = (x-2)(x-5)$. It follows that

$$p(x) = (x-1)(x-2)(x-5).$$

Example

We now divide the polynomial

$$ax^3 + bx^2 + cx + d$$

by the polynomial

$$x - r,$$

where the coefficients a, b, c, d and r of these polynomials are numbers (which may be real or complex).

The calculation may be set out as a division calculation as follows:

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & ax^2 & +\,(ar+b)x & +\,(ar^2+br+c) \\ \cline{2-5}
x-r & ax^3 & +\,bx^2 & +\,cx & +\,d \\
 & ax^3 & -\,arx^2 & & \\ \cline{2-3}
 & & (ar+b)x^2 & +\,cx & \\
 & & (ar+b)x^2 & -\,(ar^2+br)x & \\ \cline{3-4}
 & & & (ar^2+br+c)x & +\,d \\
 & & & (ar^2+br+c)x & -\,(ar^3+br^2+cr) \\ \cline{4-5}
 & & & & ar^3+br^2+cr+d
\end{array}
$$

This calculation scheme yields the result that

$$\begin{aligned} & ax^3 + bx^2 + cx + d \\ & \quad = \quad q(x)(x - r) + ar^3 + br^2 + cr + d, \end{aligned}$$

where

$$q(x) = ax^2 + (ar + b)x + (ar^2 + br + c).$$

The following lemma establishes the result more formally, using standard algebraic notation.

Lemma 2.5

Let $p(x)$ be a polynomial of degree at most 3, given by the formula

$$p(x) = ax^3 + bx^2 + cx + d,$$

where the coefficients of this polynomial are numbers (which may be real or complex), and let r be a number (which also may be real or complex). Then

$$p(x) = (x - r)q(x) + p(r),$$

where

$$q(x) = ax^2 + (ar + b)x + ar^2 + br + c.$$

Proof

$$
\begin{aligned}
p(x) &= ax^3 + bx^2 + cx + d \\
&= a(x-r)x^2 + arx^2 + bx^2 + cx + d \\
&= a(x-r)x^2 + (ar+b)x^2 + cx + d \\
&= a(x-r)x^2 + (ar+b)(x-r)x + (ar^2+br)x + cx + d \\
&= (x-r)(ax^2 + (ar+b)x) + (ar^2+br+c)x + d \\
&= (x-r)(ax^2 + (ar+b)x) + (ar^2+br+c)(x-r) \\
&\qquad + ar^3 + br^2 + cr + d \\
&= (x-r)(ax^2 + (ar+b)x + ar^2 + br + c) + p(r) \\
&= (x-r)q(x) + p(r),
\end{aligned}
$$

as required. ∎

Theorem 2.6 (Remainder Theorem)

Let $p(x)$ be a polynomial of any degree, and let r be a number. Suppose that $q(x)$ is a polynomial and k is a number determined so that

$$p(x) = q(x)(x - r) + k.$$

Then $k = p(r)$, and thus

$$p(x) = q(x)(x - r) + p(r).$$

Proof

The result follows immediately on substituting $x = r$ in the equation $p(x) = q(x)(x - r) + k$. ∎

Theorem 2.7 (Factor Theorem)

Let $p(x)$ be a polynomial of any degree, and let r be a number. Then $x - r$ is a factor of $p(x)$ if and only if $p(r) = 0$.

Proof

If $x - r$ is a factor of $p(x)$ then it follows directly that $p(r) = 0$.

Conversely suppose that $p(r) = 0$. We must prove that $x - r$ is a factor of $p(r)$. Now the Remainder Theorem ensures the existence of a polynomial $q(x)$ such that $p(x) = (x - r)q(x) + p(r)$. But $p(r) = 0$. It follows that $p(x) = (x - r)q(x)$, and thus $x - r$ is a factor of $p(x)$, as required. ∎

The following proposition is useful in limiting the number of cases that need to be considered when given a cubic polynomial with integer coefficients, and it is known that the polynomial already has at least one integer root.

Proposition 2.8

Let $p(x)$ be a polynomial of degree at most 3, given by the formula

$$p(x) = ax^3 + bx^2 + cx + d,$$

where the coefficients of this polynomial are integers, and let r be a root of this polynomial that is also an integer. Then r divides d.

Proof

The integer r is a root of the polynomial $p(x)$. It follows directly from Lemma 2.5 that

$$p(x) = q(x)(x - r),$$

where

$$q(x) = ax^2 + (ar + b)x + ar^2 + br + c.$$

Equating coefficients, we find that

$$d = -(ar^2 + br + c)r.$$

Now r, a, b, c and d are all integers. It follows that $ar^2 + br + c$ is an integer, and therefore r divides d. The result follows. $\blacksquare$

Example

Consider the polynomial $p(x)$, where

$$4x^3 - 44x^2 + 127x - 105.$$

Now $105 = 3 \times 5 \times 7$, and therefore the divisors of 105 are

$$\pm 1, \ \pm 3, \ \pm 5, \ \pm 7, \ \pm 15, \ \pm 21, \ \pm 35 \text{ and } \pm 105.$$

Calculating, we find that

$$p(1) = -18, \quad p(-1) = -280,$$

$$p(3) = -12, \quad p(-3) = -990,$$

$$p(5) = -70, \quad p(-5) = -2340,$$

$$p(7) = 0, \quad p(-7) = -4522,$$

$$p(15) = 5400, \quad p(-15) = -25410,$$

$$p(21) = 20202, \quad p(-21) = -59220,$$

$$p(35) = 121940, \quad p(-35) = -229950,$$

$$p(105) = 4158630, \quad p(-105) = -5129040.$$

It follows that 7 is the only root of the polynomial $p(x)$ that is an integer.

Polynomials can always be divided by polynomials of lower degree, taking quotient and remainder. We now give an example of polynomial division that involves dividing a polynomial of degree 4 by a quadratic polynomial.

Example

We divide the polynomial $p(x)$ by $x^2 + 2x + 2$, where

$$p(x) = x^4 + 8x^3 + 27x^2 + 39x + 28.$$

The calculation can be undertaken using the following scheme:—

2. Polynomials (continued)

$$
\begin{array}{r|rrrrr}
 & & & x^2 & +6x & +13 \\ \cline{2-6}
x^2+2x+2 & x^4 & +8x^3 & +27x^2 & +39x & +28 \\
 & x^4 & +2x^3 & +2x^2 & & \\ \cline{2-4}
 & & 6x^3 & +25x^2 & +39x & \\
 & & 6x^3 & +12x^2 & +12x & \\ \cline{3-5}
 & & & 13x^2 & +27x & +28 \\
 & & & 13x^2 & +26x & +26 \\ \cline{4-6}
 & & & & x & +2
\end{array}
$$

This calculation scheme yields the result that

$$x^4 + 8x^3 + 27x^2 + 39x + 28 = (x^2 + 2x + 2)(x^2 + 6x + 13) + x + 2.$$

We may establish this result using standard algebraic notation as follows:—

$$
\begin{aligned}
p(x) &= x^4 + 8x^3 + 27x^2 + 39x + 28 \\
&= (x^2 + 2x + 2)x^2 - 2x^3 - 2x^2 + 8x^3 + 27x^2 + 39x + 28 \\
&= (x^2 + 2x + 2)x^2 + 6x^3 + 25x^2 + 39x + 28 \\
&= (x^2 + 2x + 2)x^2 + 6(x^2 + 2x + 2)x - 12x^2 - 12x \\
&\qquad + 25x^2 + 39x + 28 \\
&= (x^2 + 2x + 2)(x^2 + 6x) + 13x^2 + 27x + 28 \\
&= (x^2 + 2x + 2)(x^2 + 6x) + 13(x^2 + 2x + 2) - 26x - 26 \\
&\qquad + 27x + 28 \\
&= (x^2 + 2x + 2)(x^2 + 6x + 13) + x + 2.
\end{aligned}
$$

Thus

$$p(x) = (x^2 + 2x + 2)(x^2 + 6x + 13) + x + 2.$$

The structure of calculation in the standard scheme can also be clarified by adding redundant terms (coloured red) as follows:—

$$
\begin{array}{r|rrrrr}
 & & & x^2 & +6x & +13 \\
\cline{2-6}
x^2+2x+2 & x^4 & +8x^3 & +27x^2 & +39x & +28 \\
 & x^4 & +2x^3 & +2x^2 & \color{red}{+0x} & \color{red}{+0} \\
\cline{2-6}
 & & 6x^3 & +25x^2 & +39x & \color{red}{+28} \\
 & & 6x^3 & +12x^2 & +12x & \color{red}{+0} \\
\cline{3-6}
 & & & 13x^2 & +27x & +28 \\
 & & & 13x^2 & +26x & +26 \\
\cline{4-6}
 & & & & x & +2
\end{array}
$$