MA1S11—Calculus Portion
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2. Polynomials

2.1. Completing the Square in Quadratic Polynomials

A *quadratic polynomial* takes the form

$$ax^2 + bx + c$$

where the *coefficients* a, b and c are numbers (which may be real or complex), and $a \neq 0$.

The qualitative behaviour of a quadratic polynomial and, in particular, the roots of a quadratic polynomial can be determined through a process of "completing the square".

The process of "completing the square", one seeks numbers p and k for which

$$ax^2 + bx + c = a(x - p)^2 + k.$$

Now

$$a(x - p)^2 + k = ax^2 - 2apx + ap^2 + k.$$

On equating coefficients of corresponding powers of x, we arrive at the equations $2ap = -b$ and $ap^2 + k = c$. Solving these equations, we find that

$$p = -\frac{b}{2a} \quad \text{and} \quad k = c - ap^2 = \frac{4ac - b^2}{4a}.$$

A number r is a *root* of the polynomial $ax^2 + bx + c$ if and only if $ar^2 + br + c = 0$. A real number r is thus a root of this polynomial if and only if $a(r - p)^2 = -k$, where

$$p = -\frac{b}{2a} \quad \text{and} \quad k = \frac{4ac - b^2}{4a}.$$

Now a real or complex number w can be determined so that $w^2 = \sqrt{b^2 - 4ac}$. Then

$$-\frac{k}{a} = \frac{b^2 - 4ac}{(2a)^2} = \left(\frac{w}{2a}\right)^2.$$

This number w may then be represented in the form

$$w = \sqrt{b^2 - 4ac}.$$

A root r of the polynomial $ax^2 + bx + c$ must satisfy the equation

$$(r - p)^2 = \left(\frac{w}{2a}\right)^2.$$

It follows that

$$r - p = \pm\frac{w}{2a},$$

and thus

$$r = p \pm \frac{w}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The process of completing the square thus yields the standard formula for the roots of a quadratic polynomial, stated in the following lemma (which follows directly from the immediately preceding remarks).

Lemma 2.1

Let $ax^2 + bx + c$ be a quadratic polynomial, where the coefficients a, b and c are real or complex numbers and $a \neq 0$. Then the roots of the polynomial are given by the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Lemma 2.2

Let $x^2 + bx + c$ be a quadratic polynomial in which the coefficient of x^2 is equal to one, and let r and s be the roots of the polynomial (with $s = r$ in the case when $b^2 = 4c$). Then $r + s = -b$ and $rs = c$.

Proof

If the roots of the quadratic polynomial are r and s then

$$x^2 + bx + c = (x - r)(x - s) = x^2 - (r + s)x + rs.$$

The result follows. $\blacksquare$

Remark

The result of Lemma 2.2 can be used to check the standard formula for the roots of a quadratic polynomial presented in Lemma 2.1. Indeed a real number x satisfies $ax^2 + bx + c = 0$, where a, b and c are real or complex numbers, with $a \neq 0$, if and only if

$$x^2 + \frac{b}{a}x + \frac{c}{a}.$$

It follows from Lemma 2.2 that real numbers r and s are roots of this quadratic polynomial if and only if

$$r + s = -\frac{b}{a} \quad \text{and} \quad rs = \frac{c}{a}.$$

Let

$$r = \frac{-b+w}{2a} \quad \text{and} \quad s = \frac{-b-w}{2a},$$

where w is some real or complex number (customarily denoted by $\sqrt{b^2-4ac}$) that satisfies the equation $w^2 = b^2 - 4ac$. Then

$$r + s = -\frac{b}{a}$$

and

$$\begin{aligned} rs &= \frac{(-b+w)(-b-w)}{4a^2} = \frac{(-b)^2 - w^2}{4a^2} = \frac{b^2-w^2}{4a^2} \\ &= \frac{b^2 - (b^2-4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}. \end{aligned}$$

It follows that r and s are indeed the roots of the quadratic polynomial $ax^2 + bx + c$.

2.2. Quadratic Polynomials with Real Coefficients

We now restrict our attention to quadratic polynomials $ax^2 + bx + c$ in which the coefficients a, b and c are real numbers and $a \neq 0$. The process of completing the square then yields the equation

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Examining the structure of the formula on the right hand side of the above equation, we can deduce immediately the following result.

Lemma 2.3

Let $ax^2 + bx + c$ be a quadratic polynomial, where the coefficients a, b and c are real numbers and $a > 0$. Then

$$ax^2 + bx + c \geq \frac{4ac - b^2}{4a}.$$

Moreover

$$ax^2 + bx + c = \frac{4ac - b^2}{4a}$$

if and only if

$$x = -\frac{b}{2a}.$$

To summarize, if the coefficients a, b, c of the quadratic polynomial $ax^2 + bx + c$ are real numbers and $a > 0$, then the quadratic polynomial achieves its minimum value when $x = -b/(2a)$.

Similarly, if the coefficients a, b, c of the quadratic polynomial $ax^2 + bx + c$ are real numbers and $a < 0$, then the quadratic polynomial achieves its maximum value when $x = -b/(2a)$.

In both cases determined by the sign of the coefficient a, the minimum value (in the case $a > 0$), or maximum value (in the case $a < 0$), is equal to

$$\frac{4ac - b^2}{4a},$$

Proposition 2.4

Let a, b and c be real numbers, where $a \neq 0$. Then the sign of the quantity $b^2 - 4ac$ determines the qualitative nature of roots of the quadratic polynomial $ax^2 + bx + c$ according to the following prescription:

Case when $b^2 > 4ac$: *in this case the polynomial has two distinct real roots;*

Case when $b^2 = 4ac$: *in this case the polynomial has a repeated root at $-b/2a$.*

Case when $b^2 < 4ac$: *in this case the polynomial has two complex roots $p + iq$ and $p - iq$, where*

$$p = -\frac{b}{2a}, \quad q = \frac{4ac - b^2}{2a}, \quad i^2 = -1.$$

2.3. Polynomial Factorization Examples

We discuss examples exemplifying the use of standard methods for solving quadratic equations.

Example

We factorize the polynomial

$$x^5 - 13x^3 + 36x.$$

as a product of linear factors of the form $x - r$, where r is some root of the polynomial $p(x)$. Now

$$x^5 - 13x^3 + 36x = x(x^4 - 13x^2 + 36).$$

Moreover

$$x^4 - 13x^2 + 36 = u^2 - 13u + 36,$$

where $u = x^2$.

2. Polynomials (continued)

Applying standard methods for finding the roots of quadratic polynomials, we find that

$$u^2 - 13u + 36 = (u - 4)(u - 9).$$

(In this case, the factorization follows directly on noting that 4 and 9 are the unique numbers whose sum is 13 and whose product is 36.) It follows that

$$x^5 - 13x^3 + 36x = x(x^2 - 4)(x^2 - 9).$$

Now

$$x^2 - 4 = (x + 2)(x - 2) \quad \text{and} \quad x^2 - 9 = (x + 3)(x - 3).$$

It follows that

$$x^5 - 13x^3 + 36x = x(x - 1)(x - 2)(x + 3)(x - 3).$$

Example

Consider the problem of identifying all non-zero real numbers x that satisfy the equation

$$\frac{1}{x^2} + \frac{2}{x} = 35.$$

There are at least two methods for solving this equation.

To apply the first method, we let $u = 1/x$. Then x satisfies the given equation if and only if the corresponding non-zero real number u satisfies

$$u^2 + 2u - 35 = 0.$$

Now

$$u^2 + 2u - 35 = (u + 7)(u - 5).$$

It follows that the non-zero values of x that solve the equation

$$\frac{1}{x^2} + \frac{2}{x} = 35.$$

are

$$x = -\frac{1}{7} \quad \text{and} \quad x = \frac{1}{5}.$$

To apply the second method, we multiply both sides of the equation

$$\frac{1}{x^2} + \frac{2}{x} = 35$$

by x^2 in order to clear denominators. We find that

$$1 + 2x = x^2 \left(1 + \frac{1}{x^2} + \frac{2}{x} \right) = 35x^2.$$

It follows that a non-zero real number x satisfies the equation

$$\frac{1}{x^2} + \frac{2}{x} = 35$$

if and only if it satisfies the quadratic equation

$$35x^2 - 2x - 1 = 0.$$

From the standard quadratic formula, we see that the roots of the polynomial $35x^2 - 2x - 1$ are x_1 and x_2, where

$$x_1 = \frac{2 + \sqrt{4 + 4 \times 35}}{70} \quad \text{and} \quad x_2 = \frac{2 - \sqrt{4 + 4 \times 35}}{70}.$$

Moreover

$$\sqrt{4 + 4 \times 35} = \sqrt{4 \times 36} = \sqrt{4} \times \sqrt{36} = 2 \times 6 = 12.$$

It follows that

$$x_1 = \frac{2 + 12}{70} = \frac{14}{70} = \frac{2}{10} = \frac{1}{5},$$

and

$$x_2 = \frac{2 - 12}{70} = -\frac{10}{70} = -\frac{1}{7}.$$

We have thus found the solutions of the given equation.

Example

We now seek to determine all positive real numbers x satisfying the equation

$$x^{\frac{2}{3}} - 5x^{\frac{1}{2}} + 6x^{\frac{1}{3}} = 0.$$

Now

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad \text{and} \quad \frac{2}{3} - \frac{1}{3} = 2 \times \frac{1}{6}.$$

It follows that

$$x^{\frac{2}{3}} - 5x^{\frac{1}{2}} + 6x^{\frac{1}{3}} = x^{\frac{1}{3}}((x^{\frac{1}{6}})^2 - 5x^{\frac{1}{6}} + 6) = x^{\frac{1}{3}}(u^2 - 5u + 6),$$

where $u = x^{\frac{1}{6}}$. Now

$$u^2 - 5u + 6 = (u - 2)(u - 3).$$

It follows that a positive real number x satisfies the equation

$$x^{\frac{2}{3}} - 5x^{\frac{1}{2}} + 6x^{\frac{1}{3}} = 0.$$

if and only if either $x^{\frac{1}{6}} = 2$ or $x^{\frac{1}{6}} = 3$. Therefore the positive real numbers x that satisfy the given equation are 64 and 729.