MA1S11—Calculus Portion
School of Mathematics, Trinity College
Michaelmas Term 2016
Lecture 6 (October 6, 2016)

David R. Wilkins

1.20. Roots of Positive Real Numbers

Proposition 1.12

Given any positive real number c, and given any natural number n, there exists a unique positive real number r with the property that $r^n = c$.

We do not prove Proposition 1.12 formally here. A very formal and rigorous treatment of the real number system might establish the existence of the real number r by proving that if

$$r = \sup\{x \in \mathbb{R} \mid x > 0 \text{ and } x^n < c\},$$

(so that r is defined as the least upper bound of the specified set), then $r^n = c$. Alternatively, such a formal treatment might deduce Proposition 1.12 of a general theorem known as the Intermediate Value Theorem.

On the other hand, it is easy to prove that the positive real number r, assuming that it exists, is the only positive real number satisfying $r^n = c$. Indeed suppose that r and s are positive real numbers and that $r^n = c = s^n$. Were it the case that $r \neq s$, then either $r < s$ or $r > s$. If it were the case that $r < s$ then $r^n < s^n$ and therefore $r^n \neq s^n$, contradicting the requirement that $r^n = c = s^n$. If it were the case that $r > s$ then $r^n > s^n$ and therefore $r^n \neq s^n$, again contradicting the requirement that $r^n = c = s^n$. Thus the possibilities that $u < v$ and $u > v$ are ruled out, and the only remaining possibility is that $u = v$.

Given any positive real number c, and given any natural number n, the decimal expansion of a positive real number r satisfying $r^n = c$ may be found as follows. For each natural number k, let r_k be the largest multiple of 10^{-k} for which $r_k^n < c$. Then the number r_k is representable as a terminating decimal, with at most k decimal digits after the decimal point, and this number r_k is the unique number of this type for which both $r_k^n < c$ and $(r_k + 10^{-k})^n \geq c$. We obtain in this fashion an infinite sequence

$$r_1, r_2, r_3, r_4, \ldots$$

of terminating decimal approximations to the nth root of c.

Moreover if natural numbers k and m satisfy $k < m$, then the decimal expansions of r_k and r_m agree up to the kth decimal place. We can therefore determine the successive decimal digits of the decimal expansion of a real number r whose decimal expansion terminated after k decimal places is equal to r_k. This real number r satisfies $r^n = c$.

Example

For example, if $c = 3$ and $n = 2$ then the procedure described above yields successive decimal approximations

$$1.7, \; 1.73, \; 1.732, \; 1.7320, \; 1.73205, \ldots$$

to a real number r satisfying $r^2 = 3$. This number r is the square root $\sqrt{3}$ of 3.

Example

Also if $c = 4$ and $n = 2$ then the procedure yields successive decimal approximations

$$1.9, \; 1.99, \; 1.999, \; 1.9999, \; 1.99999, \ldots$$

to a real number r satisfying $r^2 = 4$. Of course $r = 2$.

1.21. Laws of Indices with Fractional Exponents

Let a be a positive real number, and let q be a positive integer. We define $a^{\frac{1}{q}} = \sqrt[q]{a}$, where $\sqrt[q]{a}$ denotes the unique positive real number with the property that $(\sqrt[q]{a})^q = a$.

Note that positive real numbers a and b satisfy $a^{\frac{1}{q}} = b$ if and only if $a = b^q$.

Note also that $(a^{pq})^{\frac{1}{q}} = a^p$ for all integers p and positive integers q. Indeed the definition of $(a^{pq})^{\frac{1}{q}}$ requires that $(a^{pq})^{\frac{1}{q}} = c$, where r is the unique positive real number satisfying $c^q = a^{pq}$. But $(a^p)^q = a^{pq}$ (see Proposition 1.8). $(a^{pq})^{\frac{1}{q}} = a^p$.

Lemma 1.13

Let a be a positive real number, and let p, q, r and s be positive integers. Suppose that

$$\frac{p}{q} = \frac{r}{s}.$$

Then

$$(\sqrt[q]{a})^p = (\sqrt[s]{a})^r.$$

Proof

Basic algebra ensures that $ps = rq$. Let

$$u = (\sqrt[q]{a})^p \quad \text{and} \quad v = (\sqrt[s]{a})^r.$$

It then follows from Lemma 1.3 that

$$\begin{aligned} u^{rq} &= ((\sqrt[q]{a})^p)^{rq} = ((\sqrt[q]{a}))^{prq} = ((\sqrt[q]{a})^q)^{pr} = a^{pr}, \\ v^{ps} &= ((\sqrt[s]{a})^r)^{ps} = ((\sqrt[s]{a})^{rps} = ((\sqrt[s]{a})^s)^{pr} = a^{pr}. \end{aligned}$$

Therefore $u^{rq} = a^{pr} = v^{ps}$. But $rq = ps$, and there exists only one positive real number x satisfying the equation $x^{rq} = a^{pr}$ (see Proposition 1.12 and the remarks that follow it). Therefore $u = v$, and thus

$$(\sqrt[q]{a})^p = (\sqrt[s]{a})^r,$$

as required. ∎

Definition

Let a be a positive real number, and let t be a rational number. We define

$$a^t = (\sqrt[q]{a})^p,$$

where p and q are integers for which $q > 0$ and $p/q = t$.

In the case where $t > 0$ it follows from Lemma 1.13 that the value of $(\sqrt[q]{a})^p$ does not depend on the choice of p and q, provided that $p/q = t$. Therefore a^t is well-defined in this case.

In the case where $t < 0$, we can write $t = -p/q$, where p and q are positive integers and, in that case

$$a^t = \frac{1}{(\sqrt[q]{a})^p}.$$

It follows that a^t is well-defined in this case also.
And $a^0 = 1$, and thus a^t is well-defined when $t = 0$. Thus a^t is well-defined for all *rational numbers* t.

Let a be a positive real number and let p and q be integers, where $q > 0$. The definition of $a^{\frac{p}{q}}$ ensures that

$$a^{\frac{p}{q}} = (\sqrt[q]{a})^p.$$

It then follows from Proposition 1.8 that

$$(a^{\frac{p}{q}})^q = ((\sqrt[q]{a})^p)^q = (\sqrt[q]{a})^{pq} = ((\sqrt[q]{a})^q)^p = a^p.$$

It then follows from the definition of $\sqrt[q]{a^p}$ that

$$\sqrt[q]{a^p} = a^{\frac{p}{q}} = (\sqrt[q]{a})^p.$$

Proposition 1.14

Let a be a positive-zero real number and let t and u be rational numbers (which may be positive, negative or zero). Then $a^{t+u} = a^t a^u$.

Proof

Because t and u are rational numbers, and are thus representable as fractions where the numerators and denominators are integers, we can represent them as fractions over a common denominator. Therefore there exist integers p, q and r, where $q > 0$, such that $t = p/q$ and $u = r/q$. Then $t + u = (p + r)/q$. It then follows from Proposition 1.7 that

$$a^{t+u} = (\sqrt[q]{a})^{p+r} = (\sqrt[q]{a})^p (\sqrt[q]{a})^r = a^t a^u,$$

as required. ∎

Proposition 1.15

Let a be a positive-zero real number and let t and u be rational numbers (which may be positive, negative or zero). Then $a^{tu} = (a^t)^u$.

Proof

The exponents t and u are rational numbers, and therefore there exist integers p, q, r and s, where $q > 0$ and $s > 0$, such that $t = p/q$ and $u = r/s$. Then

$$a = (a^{\frac{1}{qs}})^{qs}.$$

It therefore follows from Proposition 1.8 that

$$a^t = (a^{\frac{1}{q}})^p = (((a^{\frac{1}{qs}})^{qs})^{\frac{1}{q}})^p = ((a^{\frac{1}{qs}})^s)^p = (a^{\frac{1}{qs}})^{sp},$$

and therefore

$$(a^t)^u = ((a^t)^{\frac{1}{s}})^r = (((a^{\frac{1}{qs}})^{sp})^{\frac{1}{s}})^r = ((a^{\frac{1}{qs}})^p)^r = (a^{\frac{1}{qs}})^{pr} = a^{tu},$$

as required. ∎

Proposition 1.16

Let a and b be positive real numbers. Then $(ab)^t = a^t b^t$ for all rational numbers t.

Proof

Let a and b be positive real numbers, and let t be a rational number. Then there exist integers p and q such that $q > 0$ and $t = p/q$. Then it follows from Proposition 1.9 that

$$(\sqrt[q]{a}\,\sqrt[q]{b})^q = (\sqrt[q]{a})^q\,(\sqrt[q]{b})^q = ab.$$

The definition of $\sqrt[q]{ab}$ as the unique positive real number u satisfying $u^q = ab$ then ensures that

$$\sqrt[q]{a}\,\sqrt[q]{b} = \sqrt[q]{ab}.$$

It then follows from Proposition 1.9 and the definitions of a^t, b^t and $(ab)^t$ that

$$a^t b^t = (\sqrt[q]{a})^p (\sqrt[q]{b})^p = (\sqrt[q]{a}\sqrt[q]{b})^p = (\sqrt[q]{ab})^p = (ab)^t,$$

as required. ■

Remark

Let a be a positive real number. At this stage we have defined and discussed the basic properties of a^x in all cases where x is a rational number. But what about $a^{\sqrt{2}}$ and a^{π}? How do we define a^x when x is an arbitrary real number that is not necessarily rational? This will be discussed in more depth when we come to discuss exponential and logarithm functions. We note here that, for all real numbers x, the real number a^x can be characterized as the unique real number with the property that $a^t \leq a^x \leq a^u$ for all rational numbers t and u satisfying $t \leq x \leq u$. It can then be shown that $a^{x+y} = a^x a^y$ and $(a^x)^y = a^{xy}$ for all real numbers x and y.

1.22. Summary of Laws of Indices

We have established the following results from the basic definitions:—

- if a is a real number, and if p and q are non-negative integers then $a^{p+q} = a^p a^q$ and $a^{pq} = (a^p)^q$ (see Lemma 1.4, and Lemma 1.5);
- if a is a *non-zero* real number, and if m and n are integers then $a^{m+n} = a^m a^n$ and $a^{mn} = (a^m)^n$ (see Proposition 1.7 and Proposition 1.15);
- if a is a *positive* real number, and if t and u are rational numbers then $a^{t+u} = a^t a^u$ and $a^{tu} = (a^t)^u$ (see Proposition 1.7 and Proposition 1.15);

- if a and b are real numbers, and if n is a non-negative integer, then $(ab)^n = a^n b^n$ (see Proposition 1.9);
- if a and b are non-zero real numbers, and if n is an integer, then $(ab)^n = a^n b^n$ (see Proposition 1.9);
- if a and b are positive real numbers, and if t is a rational number, then $(ab)^t = a^t b^t$ (see Proposition 1.16).

The following statement provides a summary of the laws of indices proved above that are applicable to powers of real numbers.

The "laws of indices" encapsulated in the formulae $a^{p+q} = a^p a^q$, $a^{pq} = (a^p)^q$ *and* $(ab)^p = a^p b^p$ *are valid in the following situations:—*

- *when* a *and* b *are real numbers and* p *and* q *are non-negative integers;*
- *when* a *and* b *are non-zero real numbers and* p *and* q *are integers;*
- *when* a *and* b *are positive real numbers and* p *and* q *are rational numbers.*