MA1S11—Calculus Portion
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David R. Wilkins

1.15. Least Upper Bounds

Let X be a subset of the set $\mathbb{R}$ of real numbers. A real number u is said to be an *upper bound* for the set X if $x \leq u$ for all $x \in X$. A real number s is said to be a *least upper bound* for the set X if s is an upper bound for the set X that is less than or equal to all other upper bounds for this set. Thus a real number s is an least upper bound for the set X if and only if the following two conditions are satisfied:—

(i) $x \leq s$ for all $x \in X$;

(ii) $s \leq u$ for all upper bounds u for the set X.

A subset X of the set $\mathbb{R}$ of real numbers is said to be *bounded above* if it has at least one upper bound.

A subset X of the real numbers can have at most one least upper bound. Indeed suppose that the real number s is a least upper bound for X and that the real number t is also a least upper bound for X. Then $s \leq t$, because s is less than or equal to all upper bounds for the set X. Similarly $t \leq s$, because t is less than or equal to all upper bounds for the set X. The inequalities $s \leq t$ and $t \leq s$ then together imply that $s = t$. Thus no subset of the real numbers can have more than least upper bound.

The least upper bound of a non-empty set X of real numbers, if it exists, is also referred to as the *supremum* of the set X. It is customarily denoted by $\sup X$. (The alternative notation $\mathrm{lub}\, X$ for the least upper bound of a set X is used in some mathematics textbooks.)

Example

Let

$$X = \{q \in \mathbb{Q} \mid q < 1\}.$$

(In other words, X is the set consisting of all rational numbers q that satisfy $q < 1$.) We show that $\sup X = 1$.

Now it is clear from the definition of the set X that the number 1 is an upper bound for the set X. Let s be a real number satisfying $s < 1$. Then there exists a natural number n large enough to ensure that

$$1 - \frac{1}{n} > s.$$

Now
$$1 - \frac{1}{n} = \frac{n-1}{n},$$
and thus $1 - \dfrac{1}{n}$ is a rational number. It follows that
$$s < 1 - \frac{1}{n} \quad \text{and} \quad 1 - \frac{1}{n} \in X,$$
and therefore the real number s is not an upper bound for the set X.

We conclude from what we have just shown that every upper bound u for the set X must satisfy $u \geq 1$. Therefore the number 1 is the least upper bound for the set X, and thus $\sup X = 1$.

1.16. Greatest Lower Bounds

Corresponding to the definition of the concept of a *least upper bound* (or *supremum*) $\sup X$ of a set X that is non-empty and bounded above, there is an analogous definition of the concept of a *greatest lower bound* (or *infimum*) $\inf X$ of a set X of real numbers that is non-empty and bounded below.

Let X be a subset of the set $\mathbb{R}$. A real number l is a *lower bound* for the set X if $l \leq x$ for all $x \in X$. The set X is said to be *bounded below* if there exists a lower bound for the set. A real number l is said to be a *greatest lower bound* for the set X if l is a lower bound for this set that is greater than or equal to every other lower bound for this set.

The greatest lower bound of a set X, if it exists, is also referred to as the *infimum* of the set X, and is denoted by $\inf X$. (In some mathematical textbooks the greatest lower bound of a set X of real numbers may be denoted by $\mathrm{glb}\, X$.)

1.17. Bounded Subsets of the Real Numbers

A non-empty subset X of the set $\mathbb{R}$ of real numbers is said to be *bounded* if it is bounded above and below. It follows from this definition that a subset X of $\mathbb{R}$ is bounded if and only if there exist fixed constants A and B such that $A \leq x \leq B$ for all $x \in X$.

Let X be a subset of $\mathbb{R}$ with both a least upper bound $\sup X$ and a greatest lower bound $\inf X$, and let a and b be real numbers satisfying $a \leq b$ for which $X \subset [a, b]$, where

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Then the real number a is a lower bound for the set X and therefore $a \leq \inf X$. Similarly the real number b is an upper bound for the set X, and therefore $\sup X \leq b$. It follows that

$$[\inf X, \sup X] \subset [a, b].$$

Thus the interval $[\inf X, \sup X]$ is the smallest closed interval that contains the set X.

Example

Let a and b be real numbers satisfying $a < b$. Then

$$\inf[a, b] = \inf[a, b) = \inf(a, b] = \inf(a, b) = a,$$

and

$$\sup[a, b] = \sup[a, b) = \sup(a, b] = \sup(a, b) = b.$$

where

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}, \quad [a, b) = \{x \in \mathbb{R} \mid a \leq x < b\},$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}, \quad (a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

Example

For each natural number n let

$$x_n = \frac{(n-1)^2}{n^2},$$

and let X be the set $\{x_n \mid n \in \mathbb{N}\}$ consisting of the members of the infinite sequence $x_1, x_2, x_3, x_4, \ldots$. Now

$$x_n = 1 - \frac{2}{n} + \frac{1}{n^2},$$

for all natural numbers n, and therefore

$$1 - \frac{2}{n} < x_n < 1$$

for all natural numbers n.

Now, given any real number s satisfying $s < 1$, there exists some positive integer n large enough to ensure that

$$1 - \frac{2}{n} > s,$$

and therefore the real number s cannot be an upper bound for the set X. But the number 1 is an upper bound for the set X. It follows that $\sup X = 1$. In this example there is no element of the set X that is equal to the least upper bound $\sup X$ of the set, and thus $\sup X \notin X$.

Now all elements of the set X are non-negative, and $x_1 = 0$. It follows that $\inf X = 0$. Moreover $\inf X \in X$.

Remark

The above example demonstrates that there is no general principle requiring bounded sets to contain their least upper bounds or their greatest lower bounds. Some bounded sets contain their least upper bounds; others do not. Similarly some bounded sets contain their greatest lower bounds; others do not.

1.18. The Least Upper Bound Principle

The following property is a fundamental property of the system of real numbers.

> **The Least Upper Bound Principle.** *Given any subset X of the set $\mathbb{R}$ of real numbers that is non-empty and bounded above, there exists a least upper bound for the set X.*

Let X be a set of real numbers that is non-empty and bounded above. It follows from the Least Upper Bound Principle that the set X has a least upper bound. This least upper bound is unique. It is denoted by $\sup X$, and is often referred to as the *supremum* of the set X.

1.19. Existence of Greatest Lower Bounds

A natural complement to the *Least Upper Bound Principle* is a principle asserting the existence of greatest lower bounds. It follows easily from the Least Upper Bound Principle that any subset X of the set $\mathbb{R}$ of real numbers that is non-empty and bounded below must have a greatest lower bound. This follows on "reflecting" the set about the number zero, replacing each real number x in the set by $-x$. Specifically let X be a non-empty set of real numbers that is bounded below, and let R be the set consisting of all real numbers that are of the form $-x$ for some element x of X. (Thus a real number x satisfies $x \in R$ if and only if $-x \in X$.). Then the set R is non-empty and bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound $\sup R$ for the set R. Then $-\sup R$ is a greatest lower bound for the set X.

We now summarize the basic consequences of the Least Upper Bound Principle, as they apply to bounded sets.

Let X be a bounded set. Then there exist uniquely determined real numbers $\sup X$ and $\inf X$ that are the least upper bound and greatest lower bounds of the set X. Then $X \subset [\inf X, \sup X]$.

We recall that an interval is *closed* if and only if it contains its endpoints. In particular the interval $[\inf X, \sup X]$ is a bounded closed interval, and $X \subset [\inf X, \sup X]$.

Now let $[a, b]$ be any bounded closed interval for which $X \subset [a, b]$. Then b is an uppper bound for the set X, and a is a lower bound for the set X, and therefore

$$a \leq \inf X \leq \sup X \leq b.$$

It follows that

$$X \subset [\inf X, \sup X] \subset [a, b].$$

Thus $[\inf X, \sup X]$ is the smallest closed bounded interval that contains (as a subset) the set X.

Example

Let

$$X = \{x \in \mathbb{R} \mid 0 \leq x < 1\}.$$

Then $\inf X = 0$ and $\sup X = 1$. Thus $X \subset [0, 1]$, where

$$[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}.$$

Moreover $[0, 1]$ is the smallest closed interval that contains the set X.

Example

Let

$$X = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

Then $\inf X = -\sqrt{2}$ and $\sup X = \sqrt{2}$. Thus $X \subset [-\sqrt{2}, \sqrt{2}]$. Moreover $[-\sqrt{2}, \sqrt{2}]$ is the smallest closed interval that contains the set X.

Example

Let X be the set consisting of the reciprocals

$$1, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \frac{1}{5}, \ldots$$

of the natural numbers. Then $\inf X = 0$ and $\sup X = 1$. Moreover $[0, 1]$ is the smallest closed interval that contains the set X.