MA1S11—Calculus Portion
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David R. Wilkins

1.7. The System of Integers

The whole numbers are referred to as *integers*. An integer may be positive negative or zero. The positive integers are the natural numbers. The negative integers are the numbers that take the form $-n$ for some natural number n. And of course 0 is the unique integer that is zero. The set consisting of all integers is denoted by $\mathbb{Z}$.

1.8. The System of Rational Numbers

A *rational number* is a number that can be expressed in the form p/q where both p and q are integers and $q \neq 0$. The set consisting of all rational numbers is denoted by $\mathbb{Q}$.

1.9. The System of Real Numbers

The rational numbers are not sufficient for the purposes of representing lengths in Euclidean geometry. Additional numbers such as $\sqrt{2}$ and π are required in order that lengths, angles and other physical magnitudes can be represented as "numbers". Such numbers are referred to as *irrational numbers*. The union of the sets of rational and irrational numbers is the set of *real numbers*. The set of real numbers is denoted by $\mathbb{R}$. If x and y are real numbers then so are $x + y$, $x - y$ and xy. Also x/y is a real number, provided that $y \neq 0$. Each positive real number x has a positive nth root $\sqrt[n]{x}$ that is a positive real number with the property that $(\sqrt[n]{x})^n = x$.

1.10. The System of Complex Numbers

The system of real numbers can be embedded within a larger number system whose elements are referred to as *complex numbers*. Any complex number may be represented in the form $a + bi$ where a and b are real numbers and i is a particular complex number that satisfies the equation $i^2 = -1$.

The set consisting of all complex numbers is denoted by $\mathbb{C}$.

Remark

We have briefly described the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ that represent the elements of the corresponding number systems, i.e., the systems of *natural numbers*, *integers*, *rational numbers*, *real numbers* and *complex numbers*. These sets are of particular importance in mathematics. Accordingly, once the basic concepts of set theory had taken root in the mathematical literature, it become commonplace for printers to represent these sets with letters **N**, **Z**, **Q**, **R** and **C** printed in boldface, to emphasize the fact that these letters are being used to denote the sets representing the basic number systems of mathematics. Of course mathematics lecturers, writing with chalk on blackboards, were not in a position to "print in boldface". Accordingly they distinguished the letters denoting these sets by adding extra strokes. The resulting glyphs accordingly were said to be written in "blackboard bold".

But once this notation became commonplace amongst mathematicians, when writing on blackboards or in handwritten manuscripts, they came to expect to see these special sets of numbers represented in the same fashion in print. Accordingly fonts were developed containing uppercase versions of the letters N, Z, Q, R and C, in the "blackboard bold" or "openface" style that had become commonplace amongst mathematicians.

1.11. The Laws of Indices for Integer Powers of Real Numbers

Let a be a real number. The positive integer powers a^n of a are defined such that $a^1 = a$ and $a^n = a^{n-1}a$ for all natural numbers n satisfying $n > 1$. This definition of positive integer powers of a real number a is an example of a *recursive definition* in which, for example, a^8 is defined in terms of a^7, which in turn is defined in terms of a^6, and so on.

Lemma 1.2

Let a be a real number and let p and q be natural numbers. Then $a^{p+q} = a^p a^q$.

Proof

The identity $a^{p+q} = a^p a^q$ can be proved by induction on q. The recursive definition of a^{p+1} ensures that, for fixed p, $a^{p+q} = a^p a^q$ when $q = 1$. Suppose that $a^{m+k} = a^m a^k$ for some natural number k. Then

$$a^{m+k+1} = a^{m+k} a = (a^m a^k) a = a^m (a^k a) = a^m a^{k+1}.$$

Thus if the identity $a^{m+n} = a^m a^n$ holds when $q = k$ for some natural number k, then it also holds for $q = k + 1$. It follows from the Principle of Mathematical Induction that the identity $a^{p+q} = a^p a^q$ holds for all real numbers a and for all natural numbers p and q, as required. ∎

Remark

As an alternative to the reasonably elaborate induction argument, one can simply note that the product of p copies of the real number a and q copies of that same real number a will be the product of $p + q$ copies of that number.

The strategy of proof by induction comes into its own in areas of mathematics like group theory and linear algebra (especially in considering powers of square matrices), where is is appropriate to present a formal argument that demonstrates the role of the Associative Law for multiplication (of group elements, or of square matrices) in establishing the result.

Lemma 1.3

Let a be a real number and let p and q be natural numbers. Then $a^{pq} = (a^p)^q$.

Proof

First we note that $a^p = (a^p)^1$, and thus, for fixed p, the identity $a^{pq} = (a^p)^q$ holds when $q = 1$. Suppose that this identity holds when $q = k$ for some natural number k, so that $a^{pk} = (a^p)^k$. Then, from Lemma 1.2,

$$a^{p(k+1)} = a^{pk+p} = a^{pk}a^p = (a^p)^k a^p = (a^p)^{k+1}.$$

Thus if the identity $a^{pq} = (a^p)^q$ holds for $q = k$, then it also holds for $q = k + 1$. It follows from the Principle of Mathematical Induction that the identity $a^{pk} = (a^p)^k$ holds for all real numbers a and for all natural numbers p and q. ∎

Given any real number a, we define $a^0 = 1$. With this definition, the identity $a^p = a^{p-1}a$ that defines a^p recursively for $p > 1$ is also valid when $p = 1$.

Note that $0^0 = 1$, according to the definition just adopted. (The identity $0^0 = 1$ is the standard definition of 0^0 that ensures that many formulae valid for non-zero values of a remain true when $a = 0$.)

Lemma 1.4

Let a be a real number and let p and q be non-negative integers. Then $a^{p+q} = a^p a^q$.

Proof

In the case when p and q are both positive, this follows from Lemma 1.2. Otherwise at least one of the non-negative integers p and q is zero, and the identity follows from the convention that $a^0 = 1$ for all real numbers a. ∎

Lemma 1.5

Let a be a real number and let p and q be non-negative integers. Then $a^{pq} = (a^p)^q$.

Proof

In the case when p and q are both positive, this follows from Lemma 1.3. Otherwise at least one of the non-negative integers p and q is zero, and therefore $a^{pq} = 1 = (a^p)^q$. ∎

If the real number a is non-zero then a^n is defined for negative integers n so as to ensure that if $n = -q$, where q is a natural number, then $a^n = (a^q)^{-1}$.

Lemma 1.6

Let a be a non-zero real number and let p and q be natural numbers. Then $a^{p-q} = \dfrac{a^p}{a^q}$.

Proof

The proof breaks down into three cases depending on whether $p - q$ is zero, positive or negative.

Suppose that $p - q = 0$. Then $p = q$ and therefore

$$\frac{a^p}{a^q} = \frac{a^p}{a^p} = 1 = a^0 = a^{p-q}.$$

Thus the result is true when $p - q$ is zero.

Next suppose that $p - q > 0$. It then follows from Lemma 1.2 $a^p = a^{p-q}a^q$. Rearranging this inequality, we find that

$$a^{p-q} = \frac{a^p}{a^q}.$$

Finally suppose that $p - q < 0$. Then

$$a^{p-q} = \frac{1}{a^{q-p}} = \frac{1}{\dfrac{a^q}{a^p}} = \frac{a^p}{a^q}.$$

We have therefore verified the result in all three cases determined by the sign of $p - q$. ∎

Proposition 1.7

Let a be a non-zero real number and let m and n be integers (which may be positive, negative or zero). Then $a^{m+n} = a^m a^n$.

Proof

Let a be a non-zero real number, let m and n be integers. Choose natural numbers p, q, r and s such that $m = p - q$ and $n = r - s$. Applying Lemma 1.6, we find that

$$a^m a^n = a^{p-q} a^{r-s} = \frac{a^p}{a^q} \times \frac{a^r}{a^s} = \frac{a^p a^r}{a^q a^s} = \frac{a^{p+r}}{a^{q+s}} = a^{p+r-q-s} = a^{m+n},$$

as required. $\blacksquare$

Proposition 1.8

Let a be a non-zero real number and let m and n be integers (which may be positive, negative or zero). Then $a^{mn} = (a^m)^n$.

Proof

In the cases where $m = 0$ and $n = 0$, both a^{mn} and $(a^m)^n$ are equal to 1, and therefore $a^{mn} = a^m a^n$ in these cases.

In the case where $m > 0$ and $n > 0$, the identity $a^{mn} = a^m a^n$ follows directly from Lemma 1.3.

Now let p and q be positive integers. Then

$$a^{-pq} = \frac{1}{a^{pq}} = \frac{1}{(a^p)^q} = \left(\frac{1}{a^p}\right)^q = (a^{-p})^q.$$

and

$$a^{-pq} = \frac{1}{a^{pq}} = \frac{1}{(a^p)^q} = (a^p)^{-q}.$$

Also

$$a^{pq} = (a^p)^q = \left(\frac{1}{a^p}\right)^{-q} = (a^{-p})^{-q}.$$

Substituting in $m = \pm p$ and $n = \pm q$ therefore yields the required identity in all cases where both m and n are non-zero. This completes the proof. $\blacksquare$

Proposition 1.9

Let a and b be real numbers. Then $(ab)^n = a^n b^n$ for all non-negative integers n. Moreover if a and b are both non-zero then $(ab)^n = a^n b^n$ for all integers n.

Proof

Let a and b be real numbers. The identity $(ab)^n = a^n b^n$ holds when $n = 0$ because $a^0 = 1$, $b^0 = 1$ and $(ab)^0 = 1$.

The required identity can be established for positive values of n, and for all real numbers a and b using the Principle of Mathematical Induction. Indeed the identity $(ab)^n = a^n b^n$ is true when $n = 1$.

Suppose that this identity is true when $n = k$, so that $(ab)^k = a^k b^k$. Then

$$(ab)^{k+1} = (ab)^k (ab) = a^k b^k ab = a^{k+1} b^{k+1}.$$

Thus if the identity $(ab)^n = a^n b^n$ is true when $n = k$ then it is also true when $n = k + 1$. It follows from the Principle of Mathematical Induction that $(ab)^n = a^n b^n$ for all positive integers n.

Now suppose that n is negative and that both a and b are non-zero. Let $p = -n$. Then $(ab)^p = a^p b^p$. Taking the reciprocal of both sides, we find that

$$(ab)^n = \frac{1}{(ab)^p} = \frac{1}{a^p} \times \frac{1}{b^p} = a^{-p} b^{-p} = a^n b^n.$$

This completes the proof that the identity $(ab)^n = a^n b^n$ holds for all non-zero real numbers a and b and for all integer values of n, whether they be positive, negative or zero. $\blacksquare$

1.12. Factorials and Binomial Coefficients

Definition

The *factorial* $n!$ of a positive integer n is defined by the formula

$$n! = 1 \times 2 \times 3 \times \cdots \times n.$$

It is thus the product of the positive integers from 1 to n. The factorial $0!$ of zero is defined so that $0! = 1$.

The definition of factorials ensures that $n! = (n-1)!n$ for all positive integers n.

Given non-negative integers n and r, where $0 \leq r \leq n$, the binomial coefficient $\left(\begin{array}{c} n \\ r \end{array} \right)$ is defined by the formula

$$\left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!}.$$

This definition ensures that

$$\left(\begin{array}{c} n \\ 0 \end{array} \right) = \left(\begin{array}{c} n \\ n \end{array} \right) = 1.$$

for all non-negative integers n.

Lemma 1.10

Let n and r be positive integers, where $1 \leq r \leq n$. Then

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Proof

Evaluating the right hand side, we see that

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!}.$$

Now the definition of factorials ensures that

$$\frac{1}{(r-1)!} = \frac{r}{r!} \quad \text{and} \quad \frac{1}{(n-r-1)!} = \frac{n-r}{(n-r)!}.$$

It follows that

$$\begin{aligned}\binom{n-1}{r-1}+\binom{n-1}{r} &= \frac{(n-1)!r}{r!(n-r)!}+\frac{(n-1)!(n-r)}{r!(n-r)!}\\ &= \frac{(n-1)!n}{r!(n-r)!}=\frac{n!}{r!(n-r)!}\\ &= \binom{n}{r}\end{aligned}$$

The result follows. ∎

1.13. The Binomial Theorem

Theorem 1.11 (Binomial Theorem)

Let x and y be real numbers. Then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

for all natural numbers n.

Proof

The definition of binomial coefficients ensures that the theorem is true when $n = 1$.

Suppose that the result holds for $n = k$, where k is some natural number, so that

$$(x+y)^k = \sum_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) x^{k-r} y^r.$$

Then

$$\begin{aligned} (x+y)^{k+1} &= (x+y)^k (x+y) = \sum_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) x^{k-r} y^r (x+y) \\ &= \sum_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) x^{k+1-r} y^r + \sum_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) x^{k-r} y^{r+1} \end{aligned}$$

Now, substituting in $r = j - 1$, where j ranges over integers from 1 to $k + 1$, and then relabelling j as r, we find that

$$\begin{aligned}\sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^{r+1} &= \sum_{j=1}^{k+1} \binom{k}{j-1} x^{k+1-j} y^{j} \\ &= \sum_{r=1}^{k+1} \binom{k}{r-1} x^{k+1-r} y^{r}\end{aligned}$$

Therefore

$$\begin{aligned}&(x+y)^{k+1} \\ &\quad = \sum_{r=0}^{k} \binom{k}{r} x^{k+1-r} y^{r} + \sum_{r=1}^{k+1} \binom{k}{r-1} x^{k+1-r} y^{r} \\ &\quad = x^{k+1} + y^{k+1} + \sum_{r=1}^{k} \left(\binom{k}{r} + \binom{k}{r-1} \right) x^{k+1-r} y^{r}.\end{aligned}$$

Now

$$\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$$

for all non-negative integers k and positive integers r (see Lemma 1.10). It follows that

$$\begin{aligned} (x+y)^{k+1} &= x^{k+1} + y^{k+1} + \sum_{r=1}^{k} \binom{k+1}{r} x^{k+1-r} y^r \\ &= \sum_{r=0}^{k+1} \binom{k+1}{r} x^{k+1-r} y^r. \end{aligned}$$

Thus if the identity

$$(x+y)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) x^{n-r} y^r$$

holds for $n = k$, where k is some natural number, then it also holds for $n = k + 1$. It follows from the Principle of Mathematical Induction that that this identity holds for all natural numbers n, as required. ∎

Remark

The equation

$$(x+y)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) x^r y^{n-r}$$

that encapsulates the Binomial Theorem is reproduced in the booklet *Formulae and Tables* prepared for use in examinations in Ireland (on page 20 of the 2016 edition). The validity of this equation requires that $x^0 = 1$ and $y^0 = 1$, in order that the correct terms appear in the sum for $r = 0$ and $r = n$. Thus if this equation is to be true for all real values of x and y, be they positive, negative or zero, one must adopt the definition that $0^0 = 1$.

1.14. Intervals

A subset I of the set $\mathbb{R}$ of real numbers is said to be an *interval* if all real numbers that lie between two elements of the set I themselves belong to I. This requires that if u, v and w are real numbers satisfying $u < v < w$, and if $u \in I$ and $w \in I$ then also $v \in I$.

It can be shown that, in addition to the empty set $\emptyset$ and the whole set $\mathbb{R}$ of real numbers, there are eight types of integrals. Four types of integrals are bounded and are determined by their endpoints a and b.

Given real numbers a and b satisfying $a \leq b$, we denote by $[a, b]$ the set consisting of all real numbers x that satisfy $a \leq x \leq b$.

Given real numbers a and b satisfying $a < b$, we denote by $[a, b)$ the set consisting of all real numbers x that satisfy $a \leq x < b$, we denote by $(a, b]$ the set consisting of all real numbers x that satisfy $a < x \leq b$, and we denote by (a, b) the set consisting of all real numbers x that satisfy $a < x < b$.

Thus

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \quad (a \leq b), \\
[a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \quad (a < b), \\
(a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \quad (a < b), \\
(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \quad (a < b).
\end{aligned}$$

In addition to the four types of bounded intervals just described, there are four types of unbounded intervals that do not include the whole of the set $\mathbb{R}$. An unbounded interval falling within one of these types is determined by a real number c representing an upper or lower endpoint:—

$$\begin{aligned}
[c, +\infty) &= \{x \in \mathbb{R} \mid x \geq c\}, \\
(c, +\infty) &= \{x \in \mathbb{R} \mid x > c\}, \\
(-\infty, c] &= \{x \in \mathbb{R} \mid x \leq c\}, \\
(-\infty, c) &= \{x \in \mathbb{R} \mid x < c\}.
\end{aligned}$$

Example

Let $I = [1, 7]$, $J = (2, 9)$, $K = [4, 6)$ and $L = (5, +\infty)$, so that

$$\begin{aligned} I &= \{x \in \mathbb{R} \mid 1 \leq x \leq 7\}, \\ J &= \{x \in \mathbb{R} \mid 2 < x < 9\}, \\ K &= \{x \in \mathbb{R} \mid 2 \leq x < 6\}, \\ L &= \{x \in \mathbb{R} \mid x > 5\}. \end{aligned}$$

Examining the relevant definitions, we find that

$$\begin{aligned} I \cup J &= \{x \in \mathbb{R} \mid 1 \leq x < 9\} = [1, 9), \\ I \cap J &= \{x \in \mathbb{R} \mid 2 < x \leq 7\} = (2, 7], \\ I \setminus J &= \{x \in \mathbb{R} \mid 1 \leq x \leq 2\} = [1, 2], \\ J \setminus I &= \{x \in \mathbb{R} \mid 7 < x < 9\} = (7, 9), \end{aligned}$$

Also, with $K = [4, 6)$ and $L = (5, +\infty)$, we find that

$$
\begin{aligned}
K \cup L &= \{x \in \mathbb{R} \mid x \geq 4\} = [4, \infty), \\
K \cap L &= \{x \in \mathbb{R} \mid 5 < x < 6\} = (5, 6), \\
K \setminus L &= \{x \in \mathbb{R} \mid 4 \leq x \leq 5\} = [4, 5], \\
L \setminus K &= \{x \in \mathbb{R} \mid x \geq 6\} = [6, +\infty).
\end{aligned}
$$

Also, with $I = [1, 7]$, $J = (2, 9)$ and $K = [4, 6)$, we find that

$$\begin{aligned} I \setminus K &= \{x \in \mathbb{R} \mid 1 \le x < 4 \text{ or } 6 \le x \le 7\} \\ &= [1, 4) \cup [6, 7], \\ J \setminus (I \setminus K) &= (2, 9) \setminus ([1, 4) \cup [6, 7)) \\ &= [4, 6) \cup [7, 9). \end{aligned}$$