MA1S11—Calculus Portion
School of Mathematics, Trinity College
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1. Sets and Number Systems

1.1. Sets

A *set* is a collection. The objects that belong to a set are referred to as the *elements* of the set. Those elements may for example be numbers, other sets, or other objects studied in a mathematical investigation.

We use the notation $p \in X$ to specify that an object p is an element of a set X.

We use the notation $p \notin X$ to specify that an object p is *not* an element of a set X.

When the number of elements in a set is sufficiently small, the set can be specified by listing those elements in braces.

Example
Let X denote the set consisting of the first five prime numbers. This set can be specified as follows:

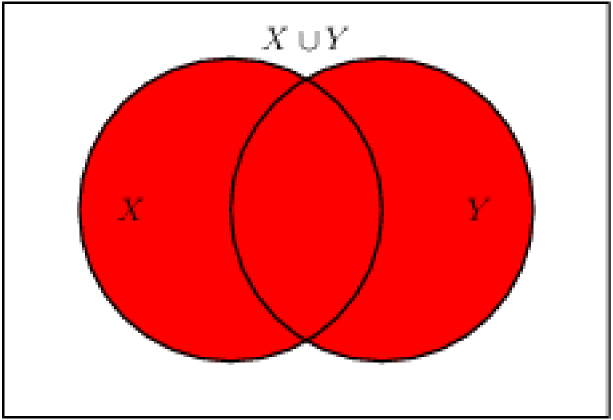
$$X = \{2, 3, 5, 7, 11\}.$$

Then $3 \in X$ and $11 \in X$. But $42 \notin X$.

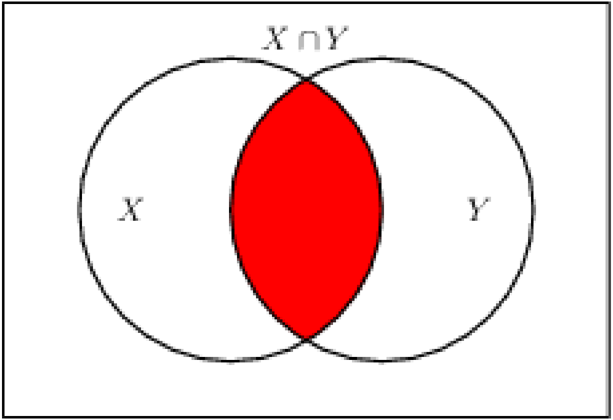
Let X and Y be sets. If the set X has the same elements as the set Y then the sets X and Y are equal (and are indeed the same set), and we may denote this by writing $X = Y$.
It follows that if sets X and Y satisfy $X \neq Y$, then either there exists an element of one of the two sets that is not an element of the other.

Example
Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{5, 4, 3, 2, 1\}$. Then $X = Y$.

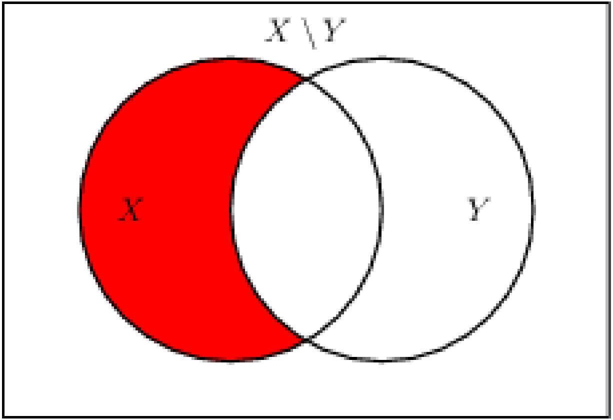
Given sets X and Y, we denote by $X \cup Y$ the *union* of the sets X and Y. This is the set consisting of those elements that belong either to X or else to Y. (This includes those elements that belong to both X and Y.)



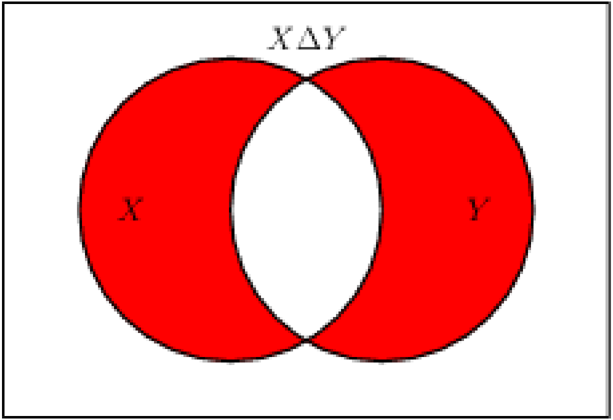
Given sets X and Y, we denote by $X \cap Y$ the *intersection* of the sets X and Y. This is the set consisting of those elements that belong to both X and Y.



Given sets X and Y, we denote by $X \setminus Y$ the *difference* of the sets X and Y. This is the set consisting of those elements that belong to the set X but not to the set Y.



The *symmetric difference* $X \Delta Y$ of sets X and Y is the union of the sets $X \setminus Y$ and $Y \setminus X$.



The definition of the symmetric difference $X \Delta Y$ of the sets X and Y ensures that

$$X \Delta Y = (X \setminus Y) \cup (Y \setminus X).$$

The symmetric difference $X \Delta Y$ of sets X and Y consists of all elements that belong to exactly one of the sets X and Y. One can verify that

$$X \Delta Y = (X \cup Y) \setminus (X \cap Y).$$

The set with no elements is referred to as the *empty set*, and is denoted by $\emptyset$.

Let X and Y be sets, and let p be an object. Then

- $p \in X \cup Y$ if and only if either $p \in X$ or $p \in Y$;
- $p \in X \cap Y$ if and only if both $p \in X$ and $p \in Y$;
- $p \in X \setminus Y$ if and only if $p \in X$ but $p \notin Y$.

Example

Let

$$X = \{2, 4, 6, 8, 10\}$$

and

$$Y = \{6, 7, 8, 9, 10\}.$$

Then

$$\begin{aligned} X \cup Y &= \{2, 4, 6, 7, 8, 9, 10\}, \\ X \cap Y &= \{6, 8, 10\}, \\ X \setminus Y &= \{2, 4\}, \\ Y \setminus X &= \{7, 9\}. \end{aligned}$$

Unions and intersections of three or more sets can be defined and represented by notation analogous to that adopted for unions and intersections of two sets. For example, if W, X, Y, and Z are sets then the union

$$W \cup X \cup Y \cup Z$$

of the sets W, X, Y and Z consists of everything that belongs to at least one of the sets W, X, Y and Z, and the intersection

$$W \cap X \cap Y \cap Z$$

of the sets W, X, Y and Z consists of everything that belongs to every one of the sets W, X, Y and Z, When constructing sets from others using the basic set operations of union, intersection and set difference, it is often necessary to specify the order of evaluation using parentheses $(\cdots)$.

Example

Let W, X, Y and Z be sets. The set

$$(W \cup X) \cap (Y \cup Z)$$

is formed by first forming the union $W \cup X$ of the sets X and W, forming also the union $Y \cup Z$ of the sets Y and Z, and then forming the intersection of the resulting sets $W \cup X$ and $Y \cap Z$.

Let X and Y be sets. If every element of the set X is an element of the set Y then we say that the set X is a *subset* of the set Y, and we denote this fact by writing $X \subset Y$.

If the set X is *not* a subset of the set Y, then we can denote this fact by writing $X \not\subset Y$.

Example

Let $X = \{1, 3, 5\}$, $Y = \{1, 2, 3, 4, 5\}$ and $Z = \{2, 3, 4, 5\}$. Then $X \subset Y$ and $Z \subset Y$, but $X \not\subset Z$.

1.2. Determining Subsets by Conditionals

Let X be a set, and let $P(x)$ represent some conditional that may or may not be satisfied by elements x of the set X. The notation

$$\{x \in X \mid P(x)\}.$$

then specifies the subset of X consisting of those elements x of the set X that satisfy the condition $P(x)$.

Example

Let

$$X = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\},$$

and let

$$W = \{x \in X \mid x^2 > 11\}.$$

Then

$$W = \{-5, -4, 4, 5\}.$$

Remark

In the mathematical literature, it is commonplace to find notation of the form

$$\{x \in X : P(x)\}$$

in place of

$$\{x \in X \mid P(x)\}.$$

The set W is thus specified in this alternative notation as follows:—

$$W = \{x \in X : x^2 > 11\}.$$

1.3. Cartesian Products of Sets

Let X and Y be sets. The *Cartesian product* $X \times Y$ of the sets X and Y is the set consisting of all ordered pairs (x, y) for which $x \in X$ and $y \in Y$.

Example

Let $X = \{1, 2, 3\}$ and $Y = \{8, 9\}$. Then

$$X \times Y = \{(1, 8), (2, 8), (3, 8), (1, 9), (2, 9), (3, 9)\}.$$

One may define in an analogous fashion the Cartesian product of three or more sets. Thus the Cartesian product $X \times Y \times Z$ of three sets X, Y and Z consists of all ordered triples (x, y, z) for which $x \in X$, $y \in Y$ and $z \in Z$.

Remark

We have considered examples of sets whose elements are numbers. Provided that the set itself is a well-defined collection of elements, the definition of *sets* imposes no restriction on what those elements might be. The elements of the set might be numbers, or characters taken from some alphabet, or strings of characters, or colours, or molecules, or students registered for a particular module. For example, the following is a perfectly valid example of a set

$$\{1, 2, 59.7, \text{‘Dog’}, \text{‘Cat’}\}.$$

1.4. The System of Natural Numbers

The positive whole numbers $1, 2, 3, 4, 5, \ldots$ are referred to as *natural numbers*. They are also known as the *positive integers*.

The set consisting of all natural numbers is denoted by $\mathbb{N}$.

1.5. Summation of Finite Sequences of Numbers

Let $T_1, T_2, T_3, \ldots$ be an infinite seqeuence of real and complex numbers, and let

$$\begin{aligned} S_1 &= T_1, \\ S_2 &= T_1 + T_2, \\ S_3 &= T_1 + T_2 + T_3, \\ &\vdots \\ S_n &= T_1 + T_2 + T_3 + \cdots + T_n. \end{aligned}$$

Then S_n is the sum of the first n members of the infinite sequence $T_1, T_2, \ldots, T_n$.

The identity

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n$$

is expressed more concisely in standard mathematical notation by writing

$$S_n = \sum_{j=1}^{n} T_j.$$

Similarly, we write

$$\sum_{j=p}^{q} T_j = T_p + T_{p+1} + \cdots + T_{q-1} + T_q.$$

We can read "$\sum\limits_{j=p}^{q} T_j$" as specifying "the sum of the quantities T_j as j ranges over all integers between p and q inclusive".

Note that any letter may be used as the "index of summation" in place of index j of summation, other than those that are already in use, or appear in formulae for the "limits of summation" above and below the $\sum$ symbol. Thus

$$\sum_{j=1}^{n} T_j = \sum_{k=1}^{n} T_k = \sum_{m=1}^{n} T_m, \quad \text{etc.}$$

Note that

$$\sum_{j=1}^{n} T_j = \sum_{k=r+1}^{r+n} T_{k-r}$$

for all integers r, because both sums involve a summation of quantities T_j with $j = 1, 2, \ldots, n$. In particular,

$$\sum_{j=0}^{n-1} T_{j+1} = \sum_{j=1}^{n} T_j = \sum_{j=2}^{n+1} T_{j-1}.$$

1.6. The Principle of Mathematical Induction

For each natural number n, let $P(n)$ be some property, in general dependent on the value of the natural number n, that must be either true or false. The Principle of Mathematical Induction asserts that the property $P(n)$ must be true for all natural numbers n provided that the following two conditions are satisfied:—

(i) $P(1)$ is true;

(ii) if $P(k)$ is true for some natural number k, then so is $P(k+1)$.

In order to illustrate the procedure for setting out a proof using the Principle of Mathematical Induction, we establish the formula stated in the following proposition, which establishes a formula for the sum of the first n terms of an arithmetic sequence.

Proposition 1.1

For each natural number n, let

$$T_n = a + (n-1)d,$$

and let

$$S_n = \sum_{j=1}^{n} T_j = T_1 + T_2 + \cdots + T_n.$$

Then

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right).$$

Proof using the Principle of Mathematical Induction
Let

$$U_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

for all natural numbers n. We must prove that $S_n = U_n$ for all natural numbers n.

Now $S_1 = T_1 = a$ and

$$U_1 = \frac{1}{2}(2a + 0 \times d) = a.$$

It follows that $S_1 = U_1$. Thus the identity $S_n = U_n$ we are seeking to prove is valid when $n = 1$. (We have now accomplished the *base step* of the induction proof.)

Now let k denote some natural number for which $S_k = U_k$, so that

$$S_k = \frac{k}{2}\left(2a + (k-1)d\right).$$

The definition of S_n for all natural numbers n ensures that $S_{k+1} = S_k + T_{k+1}$, and thus

$$S_{k+1} = S_k + a + kd.$$

The definition of U_n for all natural numbers n ensures that

$$\begin{aligned}
U_{k+1} &= \frac{k+1}{2}\left(2a + ((k+1)-1)d\right) \\
&= \frac{k}{2}(2a+kd) + \frac{1}{2}(2a+kd) \\
&= \frac{k}{2}(2a+(k-1)d) + \frac{kd}{2} + \frac{1}{2}(2a+kd) \\
&= U_k + \frac{kd}{2} + \frac{1}{2}(2a+kd) \\
&= U_k + a + kd.
\end{aligned}$$

Now k has been chosen subject to the requirement that $S_k = U_k$. It follows from the above calculations that

$$S_{k+1} = S_k + a + kd = U_k + a + kd = U_{k+1}.$$

We have thus shown that that if the identity $S_n = U_n$ holds when $n = k$, then this identity also holds for $n = k + 1$. (We have thus completed the *inductive step* of the induction proof.)
It now follows from the Principle of Mathematical Induction that $S_n = U_n$ for all natural numbers n, as required. ∎

Second Proof of Proposition 1.1

Let

$$U_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

for all natural numbers. Then there is always an infinite sequence

$$T_1, T_2, T_3, T_4, \ldots$$

with the property that

$$T_1 + T_2 + T_3 + \cdots + T_n = U_n$$

for all natural numbers n. We simply have to identify what the sequence $T_1, T_2, T_3, \ldots$ is whose sums satisfy the above formula.

Now if

$$T_1 + T_2 + T_3 + \cdots + T_n = U_n$$

for all natural numbers n then $T_1 = U_1$ and $T_n = U_n - U_{n-1}$ whenever $n > 1$. It follows that

$$T_1 = U_1 = \frac{1}{2}(2a - 0 \times d) = a.$$

Moreover if $n > 1$ then

$$
\begin{aligned}
T_n &= U_n - U_{n-1} \\
&= \frac{n}{2}\,(2a + (n-1)d) - \frac{n-1}{2}\,(2a + (n-2)d) \\
&= \frac{n}{2}\Big((2a - (n-1)d) - (2a - (n-2)d\Big) \\
&\qquad + \frac{1}{2}(2a + (n-2)d) \\
&= \frac{nd}{2} + \frac{1}{2}(2a + (n-2)d) = \frac{1}{2}(2a + (2n-2)d) \\
&= a + (n-1)d.
\end{aligned}
$$

Thus if $T_1, T_2, T_3, \ldots$ is the infinite sequence characterized by the property that

$$T_1 + T_2 + T_3 + \cdots + T_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

for all natural numbers n then

$$T_n = a + (n-1)d$$

for all natural numbers n. The result follows. ∎

It follows directly from Proposition 1.1 that

$$\sum_{j=1}^{n} j = \tfrac{1}{2} n(n+1),$$

for all natural numbers n, where

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n.$$

Third Proof of Proposition 1.1

Let n be a natural number. Consider the following table with two rows and n columns:—

1	2	3	4	$\cdots$	$n-1$	n
n	$n-1$	$n-2$	$n-3$	$\cdots$	2	1

Each row of the table sums to V_n, where

$$V_n = \sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + \cdots + n.$$

Moreover there are n columns, and each column sums to $n + 1$. It follows that $2V_n = n(n+1)$. because each side of this equality is equal to the sum of the numbers appearing as entries in the table. Dividing by 2, we find that

$$\sum_{j-1}^{n} j = \tfrac{1}{2} n(n+1).$$

1. Sets and Number Systems (continued)

Now let $T_n = a + (n-1)d$, where a and d are the initial value and increment respectively of the arithmetic sequence, and let

$$S_n = \sum_{j=1}^{n} T_j = T_1 + T_2 + T_3 + \cdots + T_n$$

for all natural numbers n. Then

$$\begin{aligned} S_n &= \sum_{j=1}^{n}(a + (j-1)d) = (a-d)n + d \times \sum_{j-1}^{n} j \\ &= (a-d)n + \frac{d}{2}n(n+1) \\ &= \frac{n}{2}\left(2(a-d) + (n+1)d\right) \\ &= \frac{n}{2}\left(2a + (n-1)d\right), \end{aligned}$$

as required. $\blacksquare$

Remark

The formula proved in Proposition 1.1 appears on page 22 of the booklet *Formulae and Tables* published by the State Examinations Commission (*Foirmlí agus Táblaí*, Coimisiún na Scrúduithe Stáit, SEC/PO 100000555-V5-Jan2016, p.22).

The Principle of Mathematical Induction had manifold applications in mathematics. In particular its application is not restricted to problems concerned with summation of sequences of numbers.

Example

We use the method of Proof by Mathematical Induction to prove that $9^n - 1$ is divisible by 8 for all natural numbers n. Now if $n = 1$ then $9^n - 1 = 8$, and thus $9^n - 1$ is divisible by 8. Thus the proposition that $9^n - 1$ is divisible by 8 holds for $n = 1$.

Suppose that this proposition holds for $n = k$, so that k is some natural number for which $9^k - 1$ is divisible by 8. Now

$$\begin{aligned} 9^{k+1} - 1 &= (9^{k+1} - 9^k) + (9^k - 1) \\ &= (9 - 1) \times 9^k + (9^k - 1) \\ &= 8 \times 9^k + (9^k - 1). \end{aligned}$$

now both 8×9^k and $9^k - 1$ are divisible by 8, and the sum of two integers divisible by 8 must itself be divisible by 8. It follows that $9^{k+1} - 1$ is divisible by 8. Thus if the proposition that $9^n - 1$ is divisible by 9 holds for $n = k$ then it also holds for $n = k + 1$. Thus the proposition that $9^n - 1$ is divisible by 8 is true for all natural numbers n, as claimed.