MA1S11—Calculus Portion
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Preview of Logarithm and Exponential Functions

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201. Preview of Logarithm and Exponential Functions

The *natural logarithm* function ln is defined so that

$$\ln s = \int_1^s \frac{1}{x}\, dx.$$

It follows from this definition, and from the Fundamental Theorem of Calculus that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Proposition 201.1

The natural logarithm function ln *satisfies*

$$\ln(uv) = \ln u + \ln v$$

for all positive real numbers u and v.

Proof

The identity

$$\int_1^{uv} \frac{1}{x}\,dx = \int_1^{u} \frac{1}{x}\,dx + \int_u^{uv} \frac{1}{x}\,dx$$

is satisfied for all positive real numbers u and v. (see Corollary 7.12). Moreover

$$\int_u^{uv} \frac{1}{x}\,dx = u\int_1^{v} \frac{1}{ux}\,dx = \int_1^{v} \frac{1}{x}\,dx = \ln v.$$

(see Proposition 7.13). It follows that

$$\ln(uv) = \ln u + \ln v,$$

as required. ∎

Let b be a positive real number satisfying $b > 1$. Then $\ln b > 0$. It follows from Proposition 201.1 that $\ln b^n = n \ln b$ for all integers n. Combining this result with the Intermediate Value Theorem, we see that the range of the logarithm function is the whole of the real line $\mathbb{R}$, and therefore, given any real number x, there exists some positive real number u for which $\ln u = x$. This real number u is uniquely-determined because the natural logarithm function is increasing.

Definition

The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is defined so that, for all real number x, $\exp(x)$ is the unique positive real number for which $\ln(\exp(x)) = x$.

Let x be a real number and let u be a positive real number. Then $x = \ln u$ if and only if $u = \exp(x)$.

We have shown that the logarithm function satisfies

$$\ln(uv) = \ln u + \ln v$$

for all positive real numbers u and v. The exponential function is the inverse of the logarithm function. It follows that

$$\exp(s+t) = \exp(s)\exp(t)$$

for all real numbers s and t.
Using this identity we can show that $\exp(qk) = \exp(k)^q$ for all rational numbers q and for all real number k. This motivates us to define $b^x = \exp((\ln b)x)$ for all real numbers x. The relevant Laws of Indices are then satisfied:

$$b^{x+y} = b^x b^y \quad \text{and} \quad b^{xy} = (b^x)^y$$

for all positive real numbers b and for all real numbers x and y. It follows that $\exp(x) = e^x$, where e is a positive constant between 2 and 3.

The exponential function satisfies

$$\frac{d}{dx}(e^{kx}) = ke^{kx}.$$

One can readily verify that if the exponential function is differentiable, then this formula must be satisfied. Indeed

$$kx = \ln(e^{kx})$$

for all real numbers k. Differentiating the identity and using the Chain Rule, we find that

$$k = \frac{1}{e^{kx}} \frac{d}{dx}\left(e^{kx}\right),$$

and therefore

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}.$$