Module MA1S11 (Calculus)
Michaelmas Term 2016
Section 9: Calculus and Motion

D. R. Wilkins

Contents

9 Calculus and Motion

9.1 Motion under a Central Force

Let the position of a particle in the plane be determined by two variables $r(t)$ and $\theta(t)$ that are functions of time that may be repeatedly differentiated any number of times, and that determine the Cartesian coordinates $(x(t), y(t))$ of the particle at time t according to the equations

$$x(t) = r(t)\,\cos\theta(t), \quad y(t) = r(t)\,\sin\theta(t).$$

Thus $r(t)$ represents the distance of the particle from the origin at time t, and $\theta(t)$ denotes the angle in radians that the line joining the particle to the origin makes with the x-axis at time t.

We denote the first and second derivatives of the quantities $r(t)$, $\theta(t)$, $x(t)$, $y(t)$ by putting dots over the respective letters so that

$$\dot{r}(t) = \frac{dr(t)}{dt}, \quad \dot{\theta}(t) = \frac{d\theta(t)}{dt},$$

$$\begin{aligned} \dot{x}(t) &= \frac{dx(t)}{dt}, \quad \dot{y}(t) = \frac{dy(t)}{dt}, \\ \ddot{r}(t) &= \frac{d^2 r(t)}{dt^2}, \quad \ddot{\theta}(t) = \frac{d^2\theta(t)}{dt^2}, \\ \ddot{x}(t) &= \frac{d^2 x(t)}{dt^2}, \quad \ddot{y}(t) = \frac{d^2 y(t)}{dt^2}. \end{aligned}$$

Now $\dot{x}(t)$ and $\dot{y}(t)$ are the Cartesian components of the velocity of the particle at time t. Differentiating we find that

$$\begin{aligned} \dot{x}(t) = \frac{dx(t)}{dt} &= \frac{d}{dt}\Big(r(t)\cos\theta(t)\Big) \\ &= \frac{dr(t)}{dt}\,\cos\theta(t) - r(t)\frac{d(\theta(t))}{dt}\,\sin\theta(t) \\ &= \dot{r}(t)\,\cos\theta(t) - r(t)\dot{\theta}(t)\,\sin\theta(t), \\ \dot{y}(t) = \frac{dy(t)}{dt} &= \frac{d}{dt}\Big(r(t)\sin\theta(t)\Big) \\ &= \frac{dr(t)}{dt}\,\sin\theta(t) + r(t)\frac{d(\theta(t))}{dt}\,\cos\theta(t) \\ &= \dot{r}(t)\,\sin\theta(t) + r(t)\dot{\theta}(t)\,\cos\theta(t). \end{aligned}$$

It is not necessary to indicate explicitly the time dependence of the quantities r, θ, x, y, $\dot{r}$, $\dot{\theta}$ $\dot{x}$ and $\dot{y}$. We may therefore simplify notation by writing the equations just derived in the form

$$\begin{aligned} \dot{x} = \frac{dx}{dt} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\ \dot{y} = \frac{dy}{dt} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \end{aligned}$$

Lemma 9.1 *Let a particle move in the plane so that the Cartesian components x and y of its position satisfy*

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Then

$$\begin{aligned} \frac{dx}{dt} &= \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta, \\ \frac{dy}{dt} &= \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta. \end{aligned}$$

We differentiate again to find the Cartesian components $\ddot{x}(t)$ and $\ddot{y}(t)$ of the acceleration of the particle in terms of $r(t)$, $\theta(t)$ and their first and second derivatives. We find that

$$\begin{aligned} \ddot{x} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) \\ &= \frac{d}{dt}(\dot{r}\cos\theta) - \frac{d}{dt}\left(r\dot{\theta}\sin\theta\right) \\ &= \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta \\ &\qquad - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\sin\theta \end{aligned}$$

Similarly

$$\begin{aligned} \ddot{y} &= \frac{d}{dt}\left(\frac{dy}{dt}\right) \\ &= \frac{d}{dt}(\dot{r}\sin\theta) + \frac{d}{dt}\left(r\dot{\theta}\cos\theta\right) \\ &= \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta \\ &\qquad + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos\theta \end{aligned}$$

We now suppose that the acceleration of the particle is always directed towards or away from the origin, and that its magnitude is determined by the distance of the particle from the origin. Thus we suppose that, when the particle is located at a distance r from the origin, its acceleration has magnitude $|g(r)|$, where $g(r)$ is a function of r defined for positive real numbers r, and is directed towards the origin when $g(r) > 0$, and away from the origin when $g(r) < 0$. Then

$$\ddot{x}(t) = -g(r(t)) \cos \theta(t), \quad \ddot{y}(t) = -g(r(t)) \sin \theta(t),$$

Simplifying notation by suppressing explicit reference to the time-dependence of the quantities involved, we find that

$$\begin{aligned} -g(r) \cos \theta = \ddot{x} &= \left(\ddot{r} - r\dot{\theta}^2\right) \cos \theta - \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) \sin \theta, \\ -g(r) \sin \theta = \ddot{y} &= \left(\ddot{r} - r\dot{\theta}^2\right) \sin \theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) \cos \theta. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= \ddot{x} \sin \theta - \ddot{y} \cos \theta = -\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)(\sin^2 \theta + \cos^2 \theta) \\ &= -\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right), \\ -g(r) &= \ddot{x} \cos \theta + \ddot{y} \sin \theta \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)(\sin^2 \theta + \cos^2 \theta) \\ &= \ddot{r} - r\dot{\theta}^2. \end{aligned}$$

We conclude therefore that

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad \text{and} \quad \ddot{r} - r\dot{\theta}^2 = -g(r).$$

We summarize the some of the main results obtained so far in the following proposition.

Proposition 9.2 *Let a particle move in the plane so that the Cartesian components x and y of its position satisfy*

$$x = r \cos \theta \quad \textit{and} \quad y = r \sin \theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that the acceleration of the particle satisfies the equations

$$\frac{d^2x}{dt^2} = -g(r) \cos \theta \quad \textit{and} \quad \frac{d^2y}{dt^2} = -g(r) \sin \theta$$

at all times, where $g(r)$ is a function of r. Then

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0 \quad \textit{and} \quad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -g(r).$$

Corollary 9.3 *Let a particle move in the plane so that the Cartesian components x and y of its position satisfy*

$$x = r\,\cos\theta \quad \textit{and} \quad y = r\,\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that the acceleration of the particle satisfies the equations

$$\frac{d^2x}{dt^2} = -g(r)\cos\theta \quad \textit{and} \quad \frac{d^2y}{dt^2} = -g(r)\sin\theta$$

at all times, where $g(r)$ is a function of r. Then

$$r^2\frac{d\theta}{dt} = h \quad \textit{and} \quad \frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -g(r),$$

where h is a constant (having the same value at all times).

Proof Differentiating, and applying the result of Proposition 9.2

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2} = r\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) = 0.$$

It follows that

$$r^2\frac{d\theta}{dt} = h$$

where h is a constant. (In particular the value of h is fixed for all times.) It follows from Proposition 9.2 that

$$-g(r) = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2r}{dt^2} - \frac{h^2}{r^3},$$

as required. ∎

There is a standard method for solving equations of motion for particle moving in the plane, attributed to Jacques Philippe Marie Binet (1786–1856), where the acceleration is directed towards a fixed point located at the origin of a Cartesian coordinate system, and is determined by the distance of the particle from that fixed point, which has been handed down the generations.

We suppose that the distance r of the particle from the origin is expressible as a function of the angle θ, at least over sufficiently short periods of time, and set

$$r(t) = \frac{1}{u(\theta(t))},$$

where $u(\theta)$ is a function of θ whose values are positive. Then

$$\begin{aligned} \frac{dr}{dt} &= -\frac{1}{(u(\theta))^2}\frac{d}{dt}\Big(u(\theta(t))\Big) \\ &= -r^2\frac{du}{d\theta}\frac{d\theta}{dt} \\ &= -h\frac{du}{d\theta}, \end{aligned}$$

where $h = r^2\dot{\theta}$. Now h is a constant whose value is the same at all times (Corollary 9.3). Thus if we differentiate again, we find that

$$\begin{aligned} \frac{d^2r}{dt^2} &= -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} \\ &= -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}. \end{aligned}$$

It then follows from Corollary 9.3 that

$$-g\left(\frac{1}{u}\right) = -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3.$$

It follows that

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2u^2}\,g\left(\frac{1}{u}\right).$$

This equation is referred to as the *Binet equation.*

We summarize the result just obtained in a proposition.

Proposition 9.4 *Let a particle move in the plane so that the Cartesian components x and y of its position satisfy*

$$x = r\cos\theta \quad \textit{and} \quad y = r\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that

$$\frac{d^2x}{dt^2} = -g(r)\cos\theta \quad \textit{and} \quad \frac{d^2y}{dt^2} = -g(r)\sin\theta$$

at all times, where $g(r)$ is a positive function of r. Then there exists a constant h such that

$$\frac{d\theta}{dt} = hu^2 \quad \textit{and} \quad \frac{d^2u}{d\theta^2} + u = \frac{1}{h^2u^2}\, g\left(\frac{1}{u}\right),$$

where $u = 1/r$.

Corollary 9.5 *Let a particle move in the plane so that the Cartesian components x and y of its position satisfy*

$$x = r\,\cos\theta \quad \textit{and} \quad y = r\,\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that

$$\frac{d^2x}{dt^2} = -\frac{\mu}{r^2}\cos\theta \quad \textit{and} \quad \frac{d^2y}{dt^2} = -\frac{\mu}{r^2}\sin\theta$$

at all times, where μ is a positive constant. Then there exists a constant h such that

$$\frac{d\theta}{dt} = hu^2 \quad \textit{and} \quad \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

where $u = 1/r$.

Theorem 9.6 *Let a particle move in the plane so that the Cartesian components x and y of its position satisfy*

$$x = r\,\cos\theta \quad \textit{and} \quad y = r\,\sin\theta$$

at all times t, where r and θ are functions of time t that may be repeatedly differentiated any number of times. Suppose also that

$$\frac{d^2x}{dt^2} = -\frac{\mu}{r^2}\cos\theta \quad \textit{and} \quad \frac{d^2y}{dt^2} = -\frac{\mu}{r^2}\sin\theta$$

at all times, where μ is a positive constant. Then there exist constants h, θ_0 and e, where $e \geq 0$, such that

$$r^2\frac{d\theta}{dt} = h \quad \textit{and} \quad \frac{h^2}{\mu r} = 1 + e\,\cos(\theta - \theta_0).$$

Proof It follows from Corollary 9.5 that there exists a constant h such that

$$\frac{d\theta}{dt} = hu^2 \quad \text{and} \quad \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

where $u = 1/r$. Let

$$w = \frac{1}{\cos\theta}\left(u - \frac{\mu}{h^2}\right)$$

throughout the time period over which the equations of motion of the particle are to be solved. Then

$$u = \frac{\mu}{h^2} + w\,\cos\theta$$

and

$$\frac{d^2u}{d\theta^2} = \left(\frac{d^2w}{d\theta^2} - w\right)\cos\theta - 2\frac{dw}{d\theta}\sin\theta,$$

and therefore

$$\frac{d^2w}{d\theta^2}\,\cos\theta - 2\frac{dw}{d\theta}\,\sin\theta = 0.$$

But then

$$\frac{d}{d\theta}\left(\frac{dw}{d\theta}\cos^2\theta\right) = \frac{d^2w}{d\theta^2}\cos^2\theta - 2\frac{dw}{d\theta}\,\cos\theta\,\sin\theta = 0.$$

It follows that

$$\frac{dw}{d\theta}\,\cos^2\theta = B,$$

where B is a constant. Thus

$$\frac{dw}{d\theta} = \frac{B}{\cos^2\theta} = \frac{d}{d\theta}\left(B\tan\theta\right).$$

It follows that $w = A + B\tan\theta$, where A and B are constants. But then

$$\frac{1}{r} = u = \frac{\mu}{h^2} + A\,\cos\theta + B\sin\theta.$$

Let

$$e = \frac{h^2\sqrt{A^2 + B^2}}{\mu}.$$

Then the point

$$\left(\frac{h^2 A}{e\mu}, \frac{h^2 B}{e\mu}\right)$$

lies on the circle of radius 1 about the origin in the plane, and therefore there exists some real number θ_0 such that

$$\frac{h^2 A}{e\mu} = \cos\theta_0 \quad \text{and} \quad \frac{h^2 B}{e\mu} = \sin\theta_0.$$

Then

$$\frac{h^2}{\mu r} = 1 + e\,\cos\theta\,\cos\theta_0 + e\,\sin\theta\,\sin\theta_0 = 1 + e\cos(\theta - \theta_0),$$

as required. ∎

For simplicity, we can orient the Cartesian coordinate system so that $\theta_0 = 0$. We also let $\ell = h^2/\mu$. Then the equation of the orbit of the particle around the origin becomes

$$\frac{\ell}{r} = 1 + e\,\cos\theta,$$

where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Multiplying both sides of this equation by r, we find that

$$\ell = r + er\cos\theta = r + ex = \sqrt{x^2 + y^2} + ex.$$

It follows that the orbit of the particle has equation

$$x^2 + y^2 = (\ell - ex)^2.$$

Expanding the right hand side of this equation and rearranging, we find that

$$(1 - e^2)x^2 + 2e\ell x + y^2 = \ell^2.$$

The shape of the curve is then determined by the value of the constant e. There are three distinct cases: $e < 1$, $e = 1$, and $e > 1$.

The case when $e = 1$ is the simplest to analyze. In that case the equation of the orbit takes the form

$$y^2 = \ell(\ell - 2x).$$

This curve is a parabola which comes closest to the origin at the point $(\frac{1}{2}\ell, 0)$. Moreover the tangent line at this point of closest approach to the origin is the line $x = \frac{1}{2}\ell$.

Now let us restrict attention to the cases where $e \neq 1$. In those cases

$$\begin{aligned}(1 - e^2)\left(x + \frac{e\ell}{1 - e^2}\right)^2 + y^2 &= \ell^2\left(1 + \frac{e^2}{1 - e^2}\right)\\ &= \frac{\ell^2}{1 - e^2},\end{aligned}$$

and therefore

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

where

$$a = \frac{\ell}{1 - e^2}.$$

In the case when $0 \leq e < 1$ we can write the equation of the orbit in the form

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b = a\sqrt{1-e^2}$. This is the equation of an *ellipse* centred on the point $(-ae, 0)$ with *semi-major axis* equal to a and *semi-minor axis* equal to b. The quantity e that determines the shape of the ellipse is known as the *eccentricity* of the ellipse. The *semi-latus rectum* of the ellipse is equal to ℓ, where $\ell = a(1-e^2)$.

The case where $0 \leq e < 1$ characterizes a closed orbit whose distance from the origin remains bounded. The cases where $e \geq 1$ describe motion in which the particle escapes "to infinity". The following corollary therefore summarizes results we have obtained in analysing the orbit of a particle in the case where the eccentricity e of the orbit satisfies $0 \leq e < 1$.

Corollary 9.7 *Let a particle move in a closed orbit in the plane so that its acceleration is always directed towards the origin and is inversely proportional to the square of the distance from the origin, and let a Cartesian coordinate system be oriented so that points where the particle is closest to the origin when it crosses the positive x-axis. Then the orbit of the particle is an ellipse with equation*

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$. Moreover if $x = r\cos\theta$ and $y = r\sin\theta$ then r and θ satisfy the equation

$$\frac{(1-e^2)a}{r} = 1 + e\cos\theta.$$

Lemma 9.8 *Let P be a point lying on an ellipse*

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$, and let O denote the origin $(0,0)$ of the Cartesian coordinate system. Then the distance $|OP|$ between the points O and P satisfies

$$|OP| = a(1-e^2) - ex.$$

Proof The equation of the ellipse ensures that

$$\begin{aligned} y^2 &= a^2(1-e^2) - (1-e^2)(x+ae)^2 \\ &= a^2(1-e^2)^2 - 2ae(1-e^2)x - (1-e^2)x^2 \end{aligned}$$

It follows that

$$\begin{aligned} |OP|^2 &= x^2 + y^2 = e^2x^2 - 2ae(1-e^2)x + a^2(1-e^2)^2 \\ &= (a(1-e^2) - ex)^2 \end{aligned}$$

The result therefore follows on taking square roots. ∎

Lemma 9.9 *Let P be a point lying on an ellipse*

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \le e < 1$ and $b = a\sqrt{1-e^2}$, let O denote the origin $(0,0)$ of the Cartesian coordinate system and let $G = (-2ae, 0)$. Then the distances $|OP|$ and $|GP|$ of the point P from the points O and G respectively satisfy the equation

$$|OP| + |GP| = 2a.$$

Proof The ellipse is invariant under the reflection of the plane in the line $x = -ae$ that sends the point (x, y) to the point $(-x - 2ae, y)$, because

$$\frac{((-x-2ae)+ae)^2}{a^2} + \frac{y^2}{b^2} = \frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2}.$$

This reflection preserves distances and swaps the points O and G. It follows from Lemma 9.8 that

$$|GP| = a(1-e^2) - e(-x-2ae) = a(1+e^2) + ex,$$

and therefore

$$|OP| + |GP| = a(1-e^2) - ex + a(1+e^2) + ex = 2a,$$

as required. ∎

An ellipse in a plane that is not a circle is determined by two distinct points F and G of that plane, together with a positive real number a, and consists of those points P of the plane for which $|FP| + |GP| = 2a$. These points F and G are referred to as the *foci* of the ellipse. Lemma 9.9 ensures that if the ellipse is specified by an equation of the form

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \le e < 1$ and $b = a\sqrt{1-e^2}$, then the points O and G with Cartesian coordinates $(0,0)$ and $(-2ae, 0)$ respectively are the foci of the ellipse. We deduce from this the following theorem.

Theorem 9.10 *Let a particle move in a closed orbit in the plane so that its acceleration is always directed towards the origin and is inversely proportional to the square of the distance from the origin. Then the orbit of the particle is an ellipse, and the origin is located at one of the two foci of that ellipse.*

We now discuss the case where eccentricity e of the orbit of the particle is greater satisfies $e > 1$. We have already shown that the equation

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1,$$

is satisfied along the orbit of the particle. The points on the orbit of the particle therefore satisfy the equstion

$$\frac{(x+ae)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b = a\sqrt{e^2-1}$. This is the equation of a *hyperbola*. The hyperbola however has two "branches" which are separate pieces disconnected from one another. One "branch" consists of those points of the hyperbola for which $a + ex \geq a$, and the other "branch" consists of those points of the hyperbola for which $a + ex \leq -a$.

Lemma 9.11 *Let P be a point lying on an hyperbola*

$$\frac{(x+ae)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $e > 1$ and $b = a\sqrt{e^2-1}$, and let O denote the origin $(0,0)$ of the Cartesian coordinate system. Then the distance $|OP|$ between the points O and P satisfies

$$|OP| = |a(e^2-1) + ex|,$$

and therefore $|OP| = a(e^2-1) + ex$ on the branch of the hyperbola for which $x \geq -(e-1)a$, and $|OP| = -a(e^2-1) - ex$ on the branch of the hyperbola for which $x \leq -(e+1)a$.

Proof The equation of the hyperbola ensures that

$$\begin{aligned} y^2 &= (e^2-1)(x+ae)^2 - a^2(e^2-1) \\ &= a^2(e^2-1)^2 + 2ae(e^2-1)x + (e^2-1)x^2 \end{aligned}$$

It follows that

$$\begin{aligned} |OP|^2 &= x^2 + y^2 = e^2x^2 + 2ae(e^2-1)x + a^2(e^2-1)^2 \\ &= (a(e^2-1) + ex)^2 \end{aligned}$$

The result therefore follows on taking square roots. ∎

Lemma 9.12 *Let P be a point lying on a hyperbola*

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$, let O denote the origin $(0,0)$ of the Cartesian coordinate system and let $G = (-2ae, 0)$. Then the distances $|OP|$ and $|GP|$ of the point P from the points O and G respectively satisfy the equation $|GP| - |OP| = 2a$ on the branch of the hyperbola on which $x \geq -(e-1)a$, and satisfy the equation $|OP| - |GP| = 2a$ on the branch of the hyperbola on which $x \leq -(e+1)a$.

Proof It follows from Lemma 9.11 that if $x \geq (1-e)a$ then $|OP| = a(e^2 - 1) + ex$, and if P is on that branch of the hyperbola for which $x \leq -(1+e)a$ then $|OP| = -a(e^2-1) - ex$. The transformation of the plane which sends (x, y) to $(-x-2ae, y)$ interchanges the points O and G, where $G = (-2ae, 0)$, and also interchanges the two branches of the hyperbola. It follows that if P lies on the branch of the hyperbola for which $x \geq (1-e)a$ then

$$|GP| = -a(e^2-1) + ex + 2ae^2 = a(e^2+1) + ex$$

It follows that

$$|GP| - |OP| = 2a.$$

The corresponding result on the other branch of the hyperbola can then be deduced on making use of the transformation sending (x, y) to $(-x-2ae, y)$ which preserves distances and swaps the two branches of the hyperbola. ∎

Let

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

be the equation of a hyperbola, where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$, let O denote the origin $(0,0)$ of the Cartesian coordinate system and let $G = (-2ae, 0)$. The points O and G are referred to as the *foci* of the hyperbola. We associate the focus O with the branch of the hyperbola on which $x \geq -(e-1)a$, and refer to it as the *primary focus* corresponding to that branch of the hyperbola. Every line passing through the primary focus intersects the corresponding branch of the hyperbola, but lines passing through the other focus do not necessarily intersect this branch. It follows from Lemma 9.12 that the branch of the hyperbola which has O as its primary focus is the locus of points P whose distances $|OP|$ and $|GP$ from the foci O and G satisfy the equation $|GP| - |OP| = 2a$.

Suppose that a particle moves under the influence of an attractive force directed towards the point O and having magnitude μ/r^2, where μ is a constant. Then the equation determining the motion of the particle can be expressed in the form

$$r^2\frac{d\theta}{dt} = h \quad \text{and} \quad \frac{\ell}{r} = 1 + e\,\cos\theta,$$

where θ denotes the angle in radians that the line joining the point O to the particle makes with some suitably chosen fixed direction, and where e, h and ℓ are constants that satisfy the conditions $e \geq 0$ and $\ell = h^2/\mu$ (see Theorem 9.6). The quantity e determines the shape of the orbit or trajectory: the orbit is a circle if $e = 0$; the orbit is an ellipse if $0 \leq e < 1$, and the point O is situated at one of the foci of the ellipse; the trajectory is a parabola if $e = 1$; the trajectory is a branch of a hyperbola if $e > 1$, and the point O is located at the primary focus of that branch of the hyperbola.

We now consider the rate at which the area swept out by a line segment joining the fixed point O to the particle increases with time. The Cartesian components of the position of the particle at time t are $x(t)$ and $y(t)$, where

$$x(t) = r(t)\cos\theta(t) \quad \text{and} \quad y(t) = r(t)\sin\theta(t).$$

Let $P(t)$ denote the position of the particle at time t, It follows from linear algebra that, given a small increment Δt of time, the triangle with vertices O, $P(t)$ and $P(t+\Delta t)$ has area $\alpha_t(\Delta t)$, where

$$\begin{aligned}
\alpha_t(\Delta t) &= \tfrac{1}{2}\left|\begin{array}{cc} r(t)\,\cos\theta(t) & r(t+\Delta t)\,\cos\theta(t+\Delta t) \\ r(t)\,\sin\theta(t) & r(t+\Delta t)\,\sin\theta(t+\Delta t)\end{array}\right| \\
&= \tfrac{1}{2}r(t)r(t+\Delta t)\left|\begin{array}{cc} \cos\theta(t) & \cos\theta(t+\Delta t) \\ \sin\theta(t) & \sin\theta(t+\Delta t)\end{array}\right| \\
&= \tfrac{1}{2}r(t)r(t+\Delta t)\Big(\cos\theta(t)\sin\theta(t+\Delta t) \\
&\qquad - \sin\theta(t)\cos\theta(t+\Delta t)\Big) \\
&= \tfrac{1}{2}r(t)r(t+\Delta t)\sin\Big(\theta(t+\Delta t) - \theta(t)\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{\Delta t\to 0}\frac{\alpha_t(\Delta t)}{\Delta t} &= \tfrac{1}{2}r(t)\,\Big(\lim_{\Delta t\to 0} r(t+\Delta t)\Big) \\
&\qquad \times \left(\lim_{\Delta t\to 0}\frac{\sin\Big(\theta(t+\Delta t)-\theta(t)\Big)}{\Delta t}\right)
\end{aligned}$$

$$\begin{aligned}
&= \tfrac{1}{2}(r(t))^2 \, \frac{d}{d\tau} \sin(\theta(t+\tau) - \theta(t))\bigg|_{\tau=0} \\
&= \tfrac{1}{2}(r(t))^2 \cos(0) \, \frac{d}{d\tau}(\theta(t+\tau) - \theta(t))\bigg|_{\tau=0} \\
&= \tfrac{1}{2}(r(t))^2 \frac{d\theta(t)}{dt}
\end{aligned}$$

We now assume that if $A(t)$ represents the area swept out by the line joining the point O to the particle since some fixed time, and thus if $A(t+\Delta) - A(t)$ represents the increment to that area over the time interval from t to $t + \Delta t$, then

$$\lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\alpha_t(\Delta))} = 1.$$

(This will be the case if, given any positive real number ε, the trajectory of the particle lies between lines parallel to that passing through the points $P(t)$ and $P(t+\Delta t)$ and lying a distance no more than ε on either side of the line through $P(t)$ and $P(t+\Delta t)$.)

We conclude that

$$\begin{aligned}
\frac{dA(t)}{dt} &= \left(\lim_{\Delta t \to 0} \frac{A(t+\delta t) - A(t)}{\alpha_t(\Delta t)} \right) \times \left(\lim_{\Delta t \to 0} \frac{\alpha_t(\Delta t)}{\Delta t} \right) \\
&= \tfrac{1}{2}(r(t))^2 \, \frac{d\theta(t)}{dt}.
\end{aligned}$$

Now, for a particle whose acceleration is directed always towards the point O, the functions $r(t)$ and $\theta(t)$ determining its trajectory satisfy the equation

$$r(t)^2 \frac{d\theta(t)}{dt} = h$$

for some constant h (see Corollary 9.3). It follows that

$$\frac{dA(t)}{dt} = \tfrac{1}{2}h.$$

We have therefore proved the following theorem.

Theorem 9.13 *Suppose that a particle moves in the plane so as to ensure that its acceleration is always directed towards a fixed point O. Then the line segment joining O to the particle sweeps out equal areas in equal times.*

We now return to the case where the acceleration of the particle towards the fixed point O is equal to μ/r^2, where μ is a constant, and r is the distance of the particle from the point O. We suppose that the particle moves in a fixed elliptic orbit with equation

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the real numbers a, b and e satisfy $a > 0$, $0 \leq e < 1$ and $b = a\sqrt{1-e^2}$.

The area of the ellipse is then πab. Indeed the ellipse can be obtained from the unit circle by stretching by a factor a in the x-direction, stretching by a factor b in the y-direction, and then translating the centre from $(0,0)$ to $(-ae,0)$. Therefore the area of the ellipse is the area π of the unit circle successively multiplied by a and b.

It follows that the *period* T of the elliptical motion (i.e., the time taken to complete a circle is given by the equation

$$T = \frac{2\pi ab}{h} = \frac{2\pi a^2\sqrt{1-e^2}}{h}.$$

Moreover

$$\frac{h^2}{\mu} = \ell = a(1-e^2)$$

(compare Theorem 9.6 and Corollary 9.7). It follows that

$$T^2 = \frac{4\pi^2 a^4(1-e^2)}{h^2} = \frac{4\pi^2 a^4(1-e^2)}{\mu a(1-e^2)} = \frac{4\pi^2 a^3}{\mu}.$$

We have therefore proved the following theorem.

Theorem 9.14 *Let a particle in the plane move in an elliptical orbit where the acceleration of the particle is directed towards one of the foci of that ellipse and has magnitude μ/r^2, where r is the distance of the particle from that focus. Then the period of the motion satisfies the equation*

$$T = 2\pi\sqrt{\frac{a^3}{\mu}},$$

where a is the semi-major axis of the ellipse.