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Section 8: The Natural Logarithm and Exponential Functions

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8 The Natural Logarithm and Exponential Functions

8.1 The Natural Logarithm Function

Definition The *natural logarithm* function $\ln\colon (0,\infty) \to \mathbb{R}$ is defined for all positive real numbers s so that

$$\ln s = \int_1^s \frac{1}{x}\,dx.$$

It follows from this definition that if s is a real number satisfying $0 < s < 1$ then

$$\ln s = -\int_s^1 \frac{1}{x}\,dx.$$

It follows from the definition of the natural logarithm function that $\ln\colon (0,\infty) \to \mathbb{R}$ is an increasing function which satisfies $\ln(0) = 0$. In particular $\ln(x) > 0$ whenever $x > 1$, and $\ln(x) < 0$ whenever $0 < x < 1$.

Remark It is commonplace in mathematical texts to denote the natural logarithm $\ln x$ of a positive real number x by $\log x$. The natural logarithm of x is also denoted by $\log_e x$.

Proposition 8.1 *The natural logarithm function* ln *satisfies*

$$\ln(uv) = \ln u + \ln v$$

for all positive real numbers u and v.

Proof The identity

$$\int_1^{uv} \frac{1}{x}\,dx = \int_1^u \frac{1}{x}\,dx + \int_u^{uv} \frac{1}{x}\,dx$$

is satisfied for all positive real numbers u and v. (see Corollary 7.12). Moreover

$$\int_u^{uv} \frac{1}{x}\,dx = u\int_1^v \frac{1}{ux}\,dx = \int_1^v \frac{1}{x}\,dx = \ln v.$$

(see Proposition 7.13). It follows that

$$\ln(uv) = \ln u + \ln v,$$

as required. ∎

Proposition 8.2 *The logarithm function* $\ln\colon (0,\infty) \to \mathbb{R}$ *is differentiable, and*

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

for all positive real numbers x.

Proof This result follows as an immediate corollary of the Fundamental Theorem of Calculus (Theorem 7.17). ∎

Proposition 8.3 *The logarithm function* $\ln\colon (0,\infty) \to \mathbb{R}$ *satisfies*

$$\int_1^s \ln(kx)\,dx = s\ln ks - s - \ln k + 1$$

for all positive real numbers s *and* k.

Proof Differentiating $x \ln x$ using the Product Rule (Proposition 5.3), we find that

$$\frac{d}{dx}(x\ln(kx)) = \ln(kx) + 1$$

It follows that

$$\ln(kx) = \frac{d}{dx}(x\ln(kx) - x).$$

Applying Corollary 7.19, we then find that

$$\begin{aligned} \int_1^s \ln(kx)\,dx &= \int_1^s \frac{d}{dx}(x\ln(kx) - x)\,dx \\ &= [x\ln(kx) - x]_1^s \\ &= s\ln(ks) - s - \ln k + 1, \end{aligned}$$

as required. ∎

Example We determine the value of the integral

$$\int_0^s \frac{x^3}{1+x^2}\,dx$$

for all real numbers s. We apply the rule for Integration by Substitution (Proposition 7.26).

Let $u = 1 + x^2$. Then $\dfrac{du}{dx} = 2x$. Also $x^2 = u - 1$. It follows that

$$\begin{aligned}
\int_0^s \frac{x^3}{1+x^2}\,dx &= \frac{1}{2}\int_0^s \frac{(u-1)}{u}\frac{du}{dx}\,dx \\
&= \frac{1}{2}\int_{u(0)}^{u(s)} \frac{(u-1)}{u}\,du \\
&= \frac{1}{2}\int_1^{1+s^2} \left(1 - \frac{1}{u}\right)du \\
&= \frac{1}{2}\left[u - \ln u\right]_1^{1+s^2} \\
&= \tfrac{1}{2}(s^2 - \ln(1+s^2)).
\end{aligned}$$

8.2 An Infinite Series converging to the Logarithm Function

Let x be a real number satisfying $-1 < x < 1$, and let n be an positive integer. Then

$$\sum_{j=0}^{n-1}(-x)^j = 1 - x + x^2 - \cdots + (-x)^{n-1} = \frac{1-(-x)^n}{1+x}$$

(see Proposition 4.3). It follows that

$$\sum_{j=0}^{n-1}(-x)^j - \frac{1}{1+x} = -\frac{(-x)^n}{1+x},$$

and therefore

$$\left|\sum_{j=0}^{n-1}(-x)^j - \frac{1}{1+x}\right| \leq \frac{|x|^n}{1-|x|}.$$

Now let s be a real number satisfying $-1 < s < 1$. Then

$$\left|\sum_{j=0}^{n-1}(-x)^j - \frac{1}{1+x}\right| \leq \frac{|x|^n}{1-|x|} \leq \frac{|s|^n}{1-|s|}.$$

for all real numbers x satisfying $|x| \leq |s|$, and thus

$$-\frac{|s|^n}{1-|s|} \leq \sum_{j=0}^{n-1}(-x)^j - \frac{1}{1+x} \leq \frac{|s|^n}{1-|s|}$$

for all real numbers x satisfying $|x| \leq |s|$. Taking the integral over the interval from 0 to x, we find that

$$-\frac{|s|^{n+1}}{1-|s|} \leq \int_0^s \left(\sum_{j=0}^{n-1} (-x)^j - \frac{1}{1+x} \right) dx \leq \frac{|s|^{n+1}}{1-|s|}.$$

But

$$\begin{aligned} \int_0^s \left(\sum_{j=0}^{n-1} (-x)^j - \frac{1}{1+x} \right) dx &= \sum_{j=0}^{n-1} \int_0^s (-x)^j \, dx - \int_0^s \frac{1}{1+x} \, dx \\ &= \sum_{j=0}^{n-1} \frac{(-1)^j}{j+1} s^{j+1} - \int_1^{1+s} \frac{1}{u} \, du \\ &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} s^k - \ln(1+s) \end{aligned}$$

We conclude therefore that

$$-\frac{|s|^{n+1}}{1-|s|} \leq \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} s^k - \ln(1+s) \leq \frac{|s|^{n+1}}{1-|s|}$$

for all positive integers n. We have therefore proved the result stated in the following proposition.

Proposition 8.4 *Let x be a real number satisfying $-1 < x < 1$. Then*

$$-\frac{|x|^{n+1}}{1-|x|} \leq \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k - \ln(1+x) \leq \frac{|x|^{n+1}}{1-|x|}$$

for all positive integers n.

It follows from this proposition that if $-1 < x < 1$ then $\ln(1+x)$ can be represented as the sum of an infinite series as follows:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \cdots$$

We can therefore calculate $\ln(1+x)$ when $-1 < x < 1$ but summing sufficiently many terms of this infinite series. If for example $|x| \leq \frac{1}{10}$ then taking ten terms of this infinite series should suffice to calculate $\ln(1+x)$ to nine decimal places.

The values of the successive approximations to $\ln(1.1)$ computed using the infinite series can be tabulated as follows. The computation has been performed using *Python*. (The value in the 17th decimal place is affected by rounding error: $\ln(1.1) = 0.09531017980432486004\ldots$ according to *WolframAlpha*.) Successive approximations to $\ln(1.1)$:—

Sum of 1 terms of ln(1 + 0.1) series = 0.1,
Sum of 2 terms of ln(1 + 0.1) series = 0.095,
Sum of 3 terms of ln(1 + 0.1) series = 0.09533333333333334,
Sum of 4 terms of ln(1 + 0.1) series = 0.09530833333333334,
Sum of 5 terms of ln(1 + 0.1) series = 0.09531033333333334,
Sum of 6 terms of ln(1 + 0.1) series = 0.09531016666666668,
Sum of 7 terms of ln(1 + 0.1) series = 0.09531018095238097,
Sum of 8 terms of ln(1 + 0.1) series = 0.09531017970238097,
Sum of 9 terms of ln(1 + 0.1) series = 0.09531017981349207,
Sum of 10 terms of ln(1 + 0.1) series = 0.09531017980349207,
Sum of 11 terms of ln(1 + 0.1) series = 0.09531017980440117,
Sum of 12 terms of ln(1 + 0.1) series = 0.09531017980431783,
Sum of 13 terms of ln(1 + 0.1) series = 0.09531017980432552,
Sum of 14 terms of ln(1 + 0.1) series = 0.09531017980432481,
Sum of 15 terms of ln(1 + 0.1) series = 0.09531017980432488,
Sum of 16 terms of ln(1 + 0.1) series = 0.09531017980432488.

8.3 The Exponential Function

Proposition 8.5 *Let x be a real number. Then there exists a positive real number u for which* $\ln u = x$.

Proof The natural logarithm function is both increasing and continuous. Moreover

$$\ln(b^n) = n\ln(b)$$

for all positive real numbers b and for all integers n. Let b be chosen such that $b > 1$. Then, given any real number x, there exists some positive integer n large enough to ensure that

$$-n\ln b \leq x \leq n\ln b.$$

Then $\ln b^{-n} \leq x \leq \ln b^n$.

The natural logarithm function is differentiable on the interval $[b^{-n}, b^n]$ (see Proposition 8.2). It is therefore continuous on that interval. The Intermediate Value Theorem (Theorem 4.28) then guarantees the existence of a real number u satisfying $b^{-n} \leq u \leq b^n$ for which $\ln u = x$. The fact that the natural logarithm function is an increasing function on the set of positive real numbers then ensures that this positive real number u is the unique positive real number for which $\ln u = x$. This completes the proof. ∎

Definition The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is defined so that, for all real number x, $\exp(x)$ is the unique positive real number for which $\ln(\exp(x)) = x$.

It follows from the definition of the natural logarithm function that, for any real number x, $\exp(x)$ is the unique positive real number u for which

$$\int_1^u \frac{1}{t}\,dt = x.$$

Remark One can also show that, given any real number x, there exists a positive real number u satisfying $\ln u = x$ using the Least Upper Bound Principle and the definition of continuity. Indeed the Least Upper Bound Principle guarantees the existence of a positive real number u that satisfies

$$u = \sup\{z \in (0, +\infty) : \ln z \leq x\}.$$

The continuity of the natural logarithm function can then be used to rule out the possibilities that $\ln u < x$ and $\ln u > x$. It follows that the number u defined as a least upper bound as specified above must satisfy $\ln u = x$.

The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is an increasing function, because the natural logarithm function is an increasing function. The range $\exp(\mathbb{R})$ of the exponential function is the set of positive real numbers.

Lemma 8.6 *The exponential function and the natural logarithm functions satisfy the identities*

$$\ln(\exp(x)) = x \quad \text{and} \quad \exp(\ln(u)) = u$$

for all real numbers x and for all positive real numbers u.

Proof It follows from the definition of the exponential function that $\ln(\exp(x)) = x$ for all real numbers x. Let u be a positive real number, and let $x = \ln(u)$. Then

$$\ln(\exp(\ln(u))) = \ln(\exp(x)) = x = \ln(u).$$

But the logarithm function is an increasing function. It follows that $\exp(\ln(u)) = u$ (Lemma 3.4). ∎

Proposition 8.7 *The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ satisfies $\exp(u + v) = \exp(u)\exp(v)$ for all real numbers u and v.*

Proof It follows from Proposition 8.1 that

$$\ln(\exp(u)\exp(v)) = \ln(\exp(u)) + \ln(\exp(v)) = u + v.$$

But $\exp(u+v)$ is by definition the unique positive real number for which $\ln(\exp(u+v)) = u+v$. It follows that $\exp(u+v) = \exp(u)\exp(v)$, as required. ∎

Corollary 8.8 *The exponential function* $\exp\colon \mathbb{R} \to \mathbb{R}$ *satisfies* $\exp(nx) = \exp(x)^n$ *for all natural numbers* n *and for all real numbers* x. u *and* v.

Proof It follows from the definition of the natural logarithm function that $\ln(1) = 0$. It follows that $\exp(0) = 1$.

If $n > 0$ then

$$\exp((n+1)x) = \exp(nx + x) = \exp(nx)\exp(x)$$

(Proposition 8.7). A straightforward proof by induction on n therefore establishes that $\exp(nx) = (\exp(x))^n$ for all positive integers n. Also $\exp(-nx)\exp(nx) = 1$ and therefore $\exp(-nx) = (\exp(x))^{-n}$ for all positive integers n. It follows that $\exp(nx) = (\exp(x))^n$ for all integers n, as required. ∎

Corollary 8.9 *Let* b *be a positive real number. Then* $b^q = \exp(kq)$ *for all rational numbers* q*, where* $k = \ln b$.

Proof Let $q = m/n$, where m and n are integers and $n > 0$, let $s = k/n$, where $k = \ln(b)$, and let $u = \exp(s)$. Then

$$u^n = \exp(ns) = \exp(k) = \exp(\ln(b)) = b.$$

(We have here made use of both Lemma 8.6 and Corollary 8.8.) and therefore $u = b^{\frac{1}{n}}$. Applying the Laws of Indices applicable when the base is a positive real number and the exponents are rational numbers (see Proposition 1.15), we find that

$$b^q = b^{\frac{m}{n}} = u^m = \exp(s)^m = \exp(ms) = \exp\left(\frac{mk}{n}\right) = \exp(kq),$$

as required. ∎

Definition Let b be a positive real number, and let x be an irrational number. We define $b^x = \exp(kx)$, where $k = \ln b$.

Proposition 8.10 *Let* b *be a positive real number. Then* $b^x = \exp(kx)$ *for all real numbers* x*, where* $k = \ln b$.

Proof The result follows from Corollary 8.9 in the case where the real number x is rational. The result follows from the definition of b^x in the case where the real number x is irrational. The result is therefore true for all real numbers x. ∎

Proposition 8.11 *Let b be a positive real number. Then $b^{x+y} = b^x b^y$ and $b^{xy} = (b^x)^y$ for all real numbers x and y.*

Proof Let x and y be real numbers, and let $k = \ln b$. Then

$$b^{x+y} = \exp(k(x+y)) = \exp(kx + ky) = \exp(kx)\exp(ky) = b^x b^y.$$

(We have here used Proposition 8.7 and Proposition 8.10.)

Also $\ln(kx) = kx$ (Lemma 8.6), and therefore

$$(b^x)^y = (\exp(kx))^y = \exp((kx)y) = \exp(kxy) = b^{xy},$$

as required. ∎

Corollary 8.12 *The exponential function satisfies $\exp(x) = e^x$ for all real numbers x, where $e = \exp(1)$.*

Proof Let $e = \exp(1)$. Then $\ln(e) = 1$. It follows from Proposition 8.10 that $e^x = \exp(x)$ for all real numbers x, as required. ∎

Remark Numerical calculations show that

$$e = \exp(1) = 2.718281828459045\ldots$$

It can be shown that

$$\begin{aligned} \exp(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{x^n}{n!}. \end{aligned}$$

What this means in practice is that, for any real number x, the value of $\exp(x)$ can be computed to any desired degree of precision by taking sufficiently many terms of the infinite series $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$. The value of e can of course be computed by setting $x = 1$ in this infinite series. The number e satisfies the identity

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n.$$

Lemma 8.13 *The exponential function* $\exp\colon \mathbb{R} \to \mathbb{R}$ *is continuous.*

Proof Let s be a real number, and let some positive real number ε be given. Then there exist positive real numbers u and v such that $s-\varepsilon \leq u < \exp(s) < v \leq s+\varepsilon$. Let δ be the smaller of the two positive real numbers $\ln v - s$ and $s - \ln u$. If x is a real number satisfying $s-\delta < x < s+\delta$ then $\ln u < x < \ln v$, and therefore $u < \exp(s) < v$. But then $s - \varepsilon < \exp(x) < s + \varepsilon$. The result follows. ∎

Proposition 8.14 *The exponential function* $\exp\colon \mathbb{R} \to \mathbb{R}$ *is differentiable, where* $\exp(x) = e^x$ *for all real numbers* x*, and*

$$\frac{d}{dx}\left(e^x\right) = e^x$$

for all real numbers x*.*

Proof Let s be a real number, let $v = \exp(s)$, and let $G\colon (0, +\infty) \to \mathbb{R}$ be defined so that

$$G(u) = \begin{cases} \dfrac{\ln(u) - s}{u - v} & \text{if } u > 0 \text{ and } u \neq v; \\ \dfrac{1}{v} & \text{if } u = v. \end{cases}$$

Then $s = \ln(v)$, and

$$\begin{aligned} \lim_{u \to v} G(u) &= \lim_{u \to v} \frac{\ln(u) - \ln(v)}{u - v} = \left.\frac{d}{du}\left(\ln u\right)\right|_{u=v} \\ &= \frac{1}{v} = G(v). \end{aligned}$$

It follows that the function G is continuous at v. It then follows from the continuity of the exponential function at s (Lemma 8.13) that the function sending each real number x to $G(\exp(x))$ is continuous at s, and thus

$$\begin{aligned} \lim_{x \to s} \frac{\exp(x) - \exp(s)}{x - s} &= \lim_{x \to s} \frac{1}{G(\exp(x))} = \frac{1}{G(\exp s)} \\ &= \frac{1}{G(v)} = v = \exp(s). \end{aligned}$$

(Specifically these identities follow from applications of Proposition 4.26, Lemma 4.16 and Proposition 4.21.) Therefore the exponential function is differentiable at s, and

$$\left.\frac{d}{dx}\left(e^x\right)\right|_{x=s} = \left.\frac{d}{dx}(\exp(x))\right|_{x=s} = \exp(x) = e^x,$$

as required. ∎

Corollary 8.15 *Let k be a real number. Then*

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

for all real numbers x.

Proof This result follows on applying Proposition 8.14 in conjunction with the Chain Rule (Proposition 5.5). ∎

Corollary 8.16 *Let b be a positive real number. Then*

$$\frac{d}{dx}\left(b^{x}\right) = (\ln b)b^{x}$$

for all real numbers x.

Proof This result follows on combining the results of Proposition 8.10 and Corollary 8.15. ∎

Proposition 8.17 *Let x be a real variable that varies over an interval D, and let the dependent variable u be a function of x with the property that*

$$\frac{du}{dx} = k(u - B)$$

for all real values of x belonging to D, where k and B are real constants. Then

$$u = Ae^{kx} + B$$

for all real values of x belonging to D, where A is a real constant.

Proof First suppose that $u > B$ for some value of x within the interval D. It follows from the Chain Rule (Proposition 5.5) that the function u of x satisfies

$$\frac{d}{dx}\left(\ln(u - B)\right) = \frac{1}{u - B}\,\frac{du}{dx} = k.$$

It follows that $\ln(u - B) = kx + C$ throughout the interval D, where C is a real constant. But then $u - B = e^{kx+C}$ for all $x \in D$, and thus

$$u = Ae^{kx} + B$$

for all $x \in D$, where $A = e^{C}$.

The result in the case where $u < B$ for some value of x within the interval x follows on applying the result just obtained with u and B replaced by $-u$ and $-B$ respectively.

If neither of these cases apply then $u = B$ throughout D. The result follows. ∎

Proposition 8.18 *Let k and s be real numbers, where $k \neq 0$. Then*

$$\int_0^s e^{kx}\,dx = \frac{1}{k}\left(e^{ks} - 1\right).$$

Proof Applying Corollary 7.19, we find that

$$\begin{aligned} \int_0^s e^{kx}\,dx &= \frac{1}{k}\int_0^s \frac{d}{dx}\left(e^{kx}\right)\,dx = \frac{1}{k}\left[e^{kx}\right]_0^s \\ &= \frac{1}{k}\left(e^{kx} - 1\right), \end{aligned}$$

as required. ∎