# Module MA1S11 (Calculus)
# Michaelmas Term 2016
# Section 6: Trigonometric Functions and their Derivatives

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## Contents

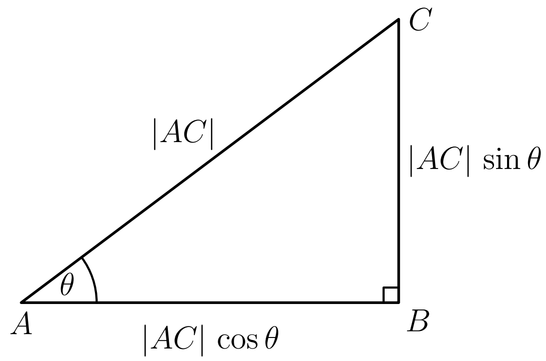
# 6 Trigonometrical Functions and their Derivatives

## 6.1 Trigonometric Functions

There are six standard trigonometric functions. They are the *sine function* (sin), the *cosine function* (cos), the *tangent function* (tan), the *cotangent function* (cot), the *secant function* (sec) and the *cosecant function* (csc).

Angles will always be represented in the following discussion using *radian measure*. If one travels a distance $s$ around a circle of radius $r$, then the angle subtended by the starting and finishing positions at the centre of the circle is $s/r$ radians.

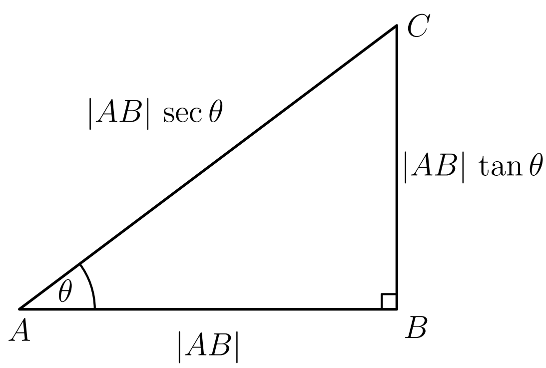
The standard trigonometrical functions represent ratios of sides of right-angled triangles, as indicated in the following diagrams.




In the above triangle $ABC$, in which the angle at the vertex $B$ is a right angle, the lengths $|AC|$, $|AB|$ and $|BC|$ satisfy the identities

$$|AB| = |AC| \cos\theta, \quad |AB| = |AC| \sin\theta,$$

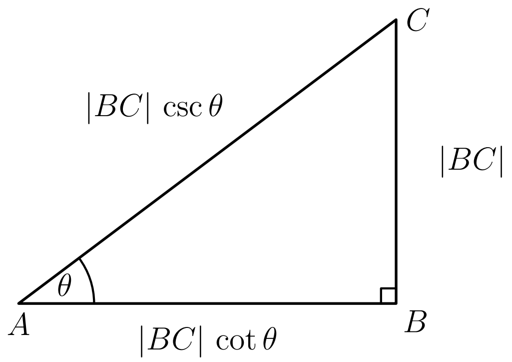
where $\theta$ denotes the angle of the triangle at the vertex $A$.



In the above triangle $ABC$, in which the angle at the vertex $B$ is a right angle, the lengths $|AC|$, $|AB|$ and $|BC|$ satisfy the identities

$$|BC| = |AB| \tan\theta, \quad |AC| = |AB| \sec\theta,$$

where $\theta$ denotes the angle of the triangle at the vertex $A$.




In the above triangle $ABC$, in which the angle at the vertex $B$ is a right angle, the lengths $|AC|$, $|AB|$ and $|BC|$ satisfy the identities

$$|AB| = |BC| \cot\theta, \quad |AC| = |BC| \csc\theta,$$

where $\theta$ denotes the angle of the triangle at the vertex $A$.

The identities described above that determine the ratios of the sides of a right angled triangle are summarized in the following proposition.

**Proposition 6.1** *Let $ABC$ be a triangle in which the angle at $B$ is a right angle, and let $\theta$ denote the angle at $A$. Then the lengths $|AB|$, $|BC|$ and $|AC|$ of the sides $AB$, $BC$ and $AC$ respectively satisfy the following identities:—*

$$|AB| = |AC| \cos\theta, \quad |BC| = |AC| \sin\theta;$$

$$|BC| = |AB| \tan\theta, \quad |AC| = |AB| \sec\theta;$$

$$|AB| = |BC| \cot\theta, \quad |AC| = |BC| \csc\theta.$$

The following trigonometrical formulae follow directly from the results stated in Proposition 6.1.

**Proposition 6.2** *The tangent, cotangent, secant and cosecant functions are determined by the sine and cosine functions in accordance with the following identities:—*

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \quad \cot\theta = \frac{\cos\theta}{\sin\theta};$$

$$\sec\theta = \frac{1}{\cos\theta}; \quad \csc\theta = \frac{1}{\sin\theta}.$$

**Proposition 6.3** *The sine and cosine functions are related by the following relationship, when angles are specified using radian measure:—*

$$\sin\theta = \cos(\tfrac{1}{2}\pi - \theta); \quad \cos\theta = \sin(\tfrac{1}{2}\pi - \theta).$$

**Proof** The trigonometrical functions are determined by ratios of edges of a right angled triangle $ABC$ in which the angle $B$ is a right angle and the angle $A$ is $\theta$ radians. The angles of a triangle add up to two right angles, and two right angles are equal to $\pi$ in radian measure. Thus if $\angle B$ denotes the angle of the right-angled triangle $ABC$ then

$$\angle A + \angle B + \angle C = \pi,$$

and thus

$$\theta + \tfrac{1}{2}\pi + \angle C = \pi,$$

and therefore $C = \frac{1}{2}\pi - \theta$. The result then follows from the definitions of the sine and cosine functions. ∎

The $n$th powers of trigonometric functions are usually presented using the following traditional notation, in instances where $n$ is a positive integer:—

$$\sin^n\theta = (\sin\theta)^n, \quad \cos^n\theta = (\cos\theta)^n, \quad \tan^n\theta = (\tan\theta)^n, \quad \text{etc.}$$

**Proposition 6.4** *The trigonometric functions satisfy the following identities:—*

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1; \\ 1 + \tan^2\theta &= \sec^2\theta; \\ 1 + \cot^2\theta &= \csc^2\theta; \end{aligned}$$

**Proof** These identities follow from the definitions of the trigonometric functions on applying Pythagoras' Theorem. ∎

## 6.2 Periodicity of the Trigonometrical Functions

Suppose that a particle moves with speed $v$ around the circumference of a circle of radius $r$, where that circle is represented in Cartesian coordinates by the equation

$$x^2 + y^2 = r^2.$$

The centre of the circle is thus at the origin of the Cartesian coordinate system. We suppose that the particle travels in an anticlockwise direction and passes through the point $(r, 0)$ when $t = 0$. Then the particle will be at the point

$$\left(r\cos\frac{vt}{r}, r\sin\frac{vt}{r}\right).$$

at time $t$. The quantities $\sin\theta$ and $\cos\theta$ are defined for all real numbers $\theta$ so that the above formula for the position of the particle moving around the circumference of the circle at a constant speed remains valid for all times.

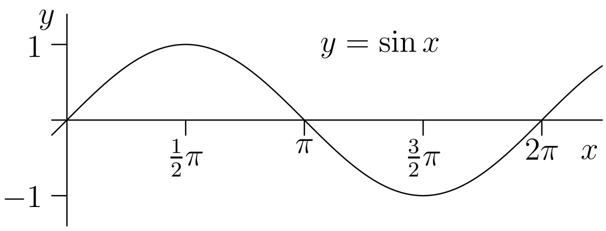
Now the particle moving round the circumference of the circle of radius $r$ with speed $v$ will complete each revolution in time $\dfrac{2\pi r}{v}$. Thus

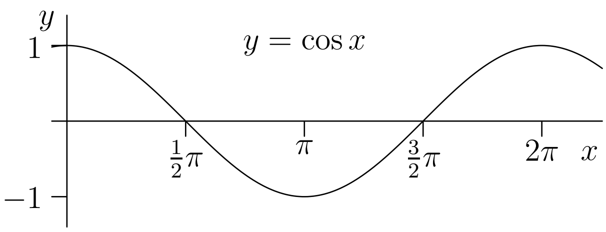
$$\cos(\theta + 2\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2\pi) = \sin\theta$$

for all real numbers $\theta$. It follows that

$$\cos(\theta + 2n\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2n\pi) = \sin\theta$$

for all real numbers $\theta$ and for all integers $n$. These equations express the *periodicity* of the sine and cosine functions.




## 6.3 Values of Trigonometric Functions at Particular Angles

The following table sets out the values of $\sin\theta$ and $\cos\theta$ for some angles $\theta$ that are multiples of $\frac{1}{2}\pi$:—

| $\theta$ | $-\pi$ | $-\frac{1}{2}\pi$ | $0$ | $\frac{1}{2}\pi$ | $\pi$ | $\frac{3}{2}\pi$ | $2\pi$ | $\frac{5}{2}\pi$ |
|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $-1$ | $0$ | $1$ | $0$ | $-1$ | $0$ | $1$ |
| $\cos\theta$ | $-1$ | $0$ | $1$ | $0$ | $-1$ | $0$ | $1$ | $0$ |

The following values of the sine and cosine functions can be derived using geometric arguments involving the use of Pythagoras' Theorem:—

| $\theta$ | $0$ | $\frac{1}{6}\pi$ | $\frac{1}{4}\pi$ | $\frac{1}{3}\pi$ | $\frac{1}{2}\pi$ |
|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| $\cos\theta$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ |

## 6.4 Addition Formulae satisfied by the Sine and Cosine Functions

We derive the standard addition formulae for trigonometric functions by considering the formulae that implement rotations of the plane about some origin chosen within that plane.

An anticlockwise rotation about the origin through an angle of $\theta$ radians sends a point $(x, y)$ of the plane to the point $(x', y')$, where

$$\left\{\begin{array}{rcl} x' &=& x\cos\theta - y\sin\theta \\ y' &=& x\sin\theta + y\cos\theta \end{array}\right.$$

(This follows easily from the fact that such a rotation takes the point $(1, 0)$ to the point $(\cos\theta, \sin\theta)$ and takes the point $(0, 1)$ to the point $(-\sin\theta, \cos\theta)$.) An anticlockwise rotation about the origin through an angle of $\varphi$ radians then sends the point $(x', y')$ of the plane to the point $(x'', y'')$, where

$$\left\{\begin{array}{rcl} x'' &=& x'\cos\varphi - y'\sin\varphi \\ y'' &=& x'\sin\varphi + y'\cos\varphi \end{array}\right.$$

Now an anticlockwise rotation about the origin through an angle of $\theta + \varphi$ radians sends the point $(x, y)$, of the plane to the point $(x'', y'')$, and thus

$$\left\{\begin{array}{rcl} x'' &=& x\cos(\theta+\varphi) - y\sin(\theta+\varphi) \\ y'' &=& x\sin(\theta+\varphi) + y\cos(\theta+\varphi) \end{array}\right.$$

But if we substitute the expressions for $x'$ and $y'$ in terms of $x$, $y$ and $\theta$ obtained previously into the above equation, we find that

$$\left\{\begin{array}{rcl} x'' &=& x(\cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi) - y(\sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi) \\ y'' &=& x(\sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi) + y(\cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi) \end{array}\right.$$

On comparing equations, we see that

$$\cos(\theta + \varphi) = \cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi,$$

and

$$\sin(\theta + \varphi) = \sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi.$$

On replacing $\varphi$ by $-\varphi$, and noting that $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -\sin\varphi$, we find that

$$\cos(\theta - \varphi) = \cos\theta\,\cos\varphi + \sin\theta\,\sin\varphi,$$

and

$$\sin(\theta - \varphi) = \sin\theta\,\cos\varphi - \cos\theta\,\sin\varphi.$$

We have therefore established the addition formulae for the sine and cosine functions stated in the following proposition.

**Proposition 6.5** *The sine and cosine functions satisfy the following identities for all real numbers $\theta$ and $\varphi$:—*

$$\begin{aligned}
\sin(\theta + \varphi) &= \sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi;\\
\sin(\theta - \varphi) &= \sin\theta\,\cos\varphi - \cos\theta\,\sin\varphi;\\
\cos(\theta + \varphi) &= \cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi;\\
\cos(\theta - \varphi) &= \cos\theta\,\cos\varphi + \sin\theta\,\sin\varphi.
\end{aligned}$$

**Remark** The equations describing how Cartesian coordinates of points of the plane transform under rotations about the origin may be written in matrix form as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Also equation (6.4) may be written

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

It follows from basic properties of matrix multiplication that

$$\begin{aligned}
&\begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix}\\
&\qquad = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.
\end{aligned}$$

Therefore

$$\begin{aligned}\cos(\theta+\varphi) &= \cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi\\ \sin(\theta+\varphi) &= \sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi.\end{aligned}$$

This provides an alternative derivation of the addition formulae stated in Proposition 6.5.

**Corollary 6.6** *The sine and cosine functions satisfy the following identities for all real numbers $\theta$:—*

$$\begin{aligned}\sin(\theta+\tfrac{1}{2}\pi) &= \cos\theta,\\ \cos(\theta+\tfrac{1}{2}\pi) &= -\sin\theta,\\ \sin(\theta+\pi) &= -\sin\theta,\\ \cos(\theta+\pi) &= -\cos\theta,\end{aligned}$$

**Proof** These results follow directly on applying Proposition 6.5 in view of the identities

$$\sin\tfrac{1}{2}\pi = 1, \quad \cos\tfrac{1}{2}\pi = 0, \quad \sin\pi = 0 \quad\text{and}\quad \cos\pi = -1. \quad \blacksquare$$

The formulae stated in the following corollary follow directly from the addition formulae stated in Proposition 6.5 on adding and subtracting those addition formulae.

**Corollary 6.7** *The sine and cosine functions satisfy the following identities for all real numbers $\theta$ and $\varphi$:—*

$$\begin{aligned}\sin\theta\,\sin\varphi &= \tfrac{1}{2}(\cos(\theta-\varphi)-\cos(\theta+\varphi));\\ \cos\theta\,\cos\varphi &= \tfrac{1}{2}(\cos(\theta+\varphi)+\cos(\theta-\varphi));\\ \sin\theta\,\cos\varphi &= \tfrac{1}{2}(\sin(\theta+\varphi)+\sin(\theta-\varphi)).\end{aligned}$$

**Corollary 6.8** *The sine and cosine functions satisfy the following identities for all real numbers $\theta$:—*

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\,\cos\theta;\\ \cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta.\end{aligned}$$

**Proof** The formula for $\sin 2\theta$ and the first formula for $\cos 2\theta$ follow from the identities stated in Proposition 6.5 on setting $\varphi = \theta$ in the formulae for $\sin(\theta + \varphi)$ and $\cos(\theta + \varphi)$. The second and third formulae for $\cos 2\theta$ then follow on making use of the identity $\sin^2\theta + cos^2\theta = 1$. ∎

The following formulae then follow directly from those stated in Corollary 6.8.

**Corollary 6.9** *The sine and cosine functions satisfy the following identities for all real numbers $\theta$:—*

$$\begin{aligned}\sin^2\theta &= \tfrac{1}{2}(1 - \cos 2\theta);\\ \cos^2\theta &= \tfrac{1}{2}(1 + \cos 2\theta).\end{aligned}$$

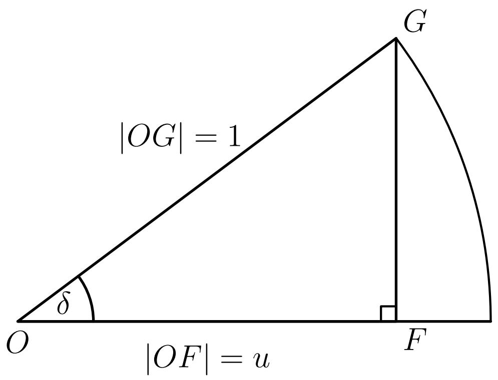
## 6.5 Derivatives of Trigonometrical Functions

**Lemma 6.10** *Let $\varepsilon$ be a positive real number. Then there exists some positive real number $\delta$ satisfying $0 < \delta < \frac{1}{2}\pi$ with the property that $1 - \varepsilon < \cos\theta < 1$ whenever $0 < \theta < \delta$.*

**Proof** Choose a real number $u$ satisfying $0 < u < 1$ for which $1 - \varepsilon \leq u$. Let a right-angled triangle $OFG$ be constructed so that the angle at $F$ is a right angle, $|OF| = u$ and $|FG| = \sqrt{1 - u^2}$, and let $\delta$ be the angle of this triangle at the vertex $O$. Then $|OG|^2 = |OF|^2 + |FG|^2 = 1$, and therefore $u = \cos\delta$. It follows that if $\theta$ is a positive real number satisfying $0 < \theta < \delta$ then

$$1 - \varepsilon \leq \cos\delta < \cos\theta < 1.$$

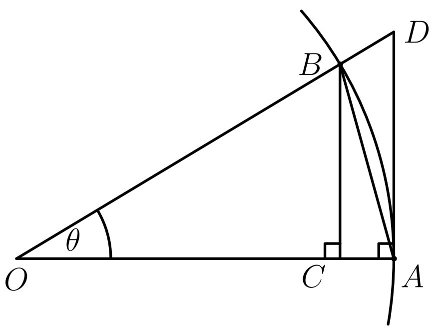
The result follows. ∎



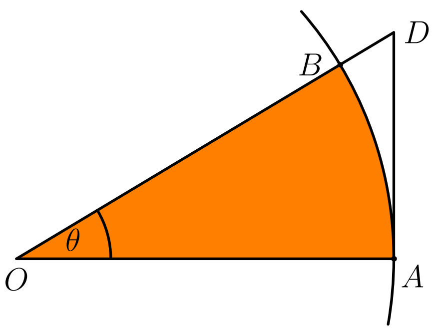
**Proposition 6.11** *Let* $\sin\colon \mathbb{R} \to \mathbb{R}$ *be the sine function whose value* $\sin\theta$*, for a given real number* $\theta$ *is the sine of an angle of* $\theta$ *radians. Then*

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

**Proof** Let a circle of unit radius pass through points $A$ and $B$, so that the angle $\theta$ in radians between the line segements $OA$ and $OB$ at the centre $O$ of the circle satisfies the inequalities $0 < \theta < \frac{1}{2}\pi$. Let $C$ be the point on the line segment $OA$ for which the angle $OCB$ is a right angle, and let the line $OB$ be produced to the point $D$ determined so that the angle $OAD$ is a right angle.




The sector $OAB$ of the unit circle is by definition the region bounded by the arc $AB$ of the circle and the radii $OA$ and $OB$. Now the area of a sector of a circle subtending at the centre an angle of $\theta$ radians is equal to the area of the circle multiplied by $\dfrac{\theta}{2\pi}$. But the area of a circle of unit radius is $\pi$. It follows that a sector of the unit circle subtending at the centre an angle of $\theta$ radians has area $\frac{1}{2}\theta$.



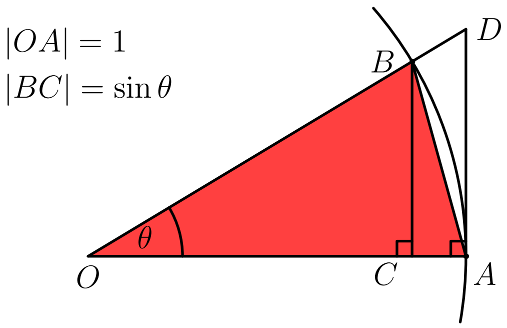

The the area of a triangle is half the base of the triangle multiplied by the height of the triangle. The base $|OA|$ and height $|BC|$ of the triangle

$AOB$ satisfy

$$|OA| = 1, \quad |BC| = \sin\theta.$$

It follows that

$$\text{area of triangle } OAB = \tfrac{1}{2} \times |OA| \times |BC| = \tfrac{1}{2}\sin\theta.$$

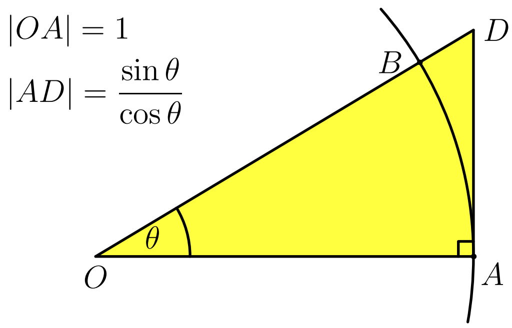



Also the base $|OA|$ and height $|AD|$ of the triangle $AOD$ satisfy

$$|OA| = 1, \quad |AD| = \frac{\sin\theta}{\cos\theta}.$$

It follows that

$$\text{area of triangle } OAD = \tfrac{1}{2} \times |OA| \times |AD| = \frac{\sin\theta}{2\cos\theta}.$$




The results concerning areas just obtained can be summarized as follows:—

$$\begin{aligned} \text{area of triangle } OAB &= \tfrac{1}{2} \times |OA| \times |BC| \\ &= \tfrac{1}{2}\sin\theta, \end{aligned}$$

$$\begin{aligned}\text{area of sector } OAB &= \frac{\theta}{2\pi} \times \pi = \tfrac{1}{2}\theta, \\ \text{area of triangle } OAD &= \tfrac{1}{2} \times |OA| \times |AD| \\ &= \tfrac{1}{2}\tan\theta = \frac{\sin\theta}{2\cos\theta}.\end{aligned}$$

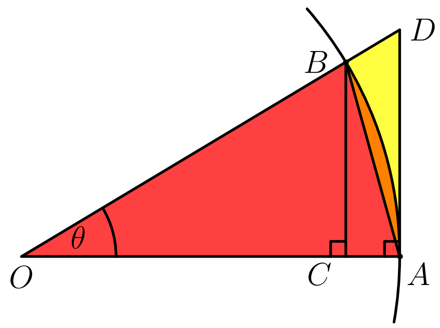
Moreover the triangle $OAB$ is strictly contained in the sector $OAB$, which in turn is strictly contained in the triangle $OAD$. It follows that

$$\text{area}(\triangle OAB) < \text{area}(\text{sector } OAB) < \text{area}(\triangle OAD),$$

and thus

$$\tfrac{1}{2}\sin\theta < \tfrac{1}{2}\theta < \frac{\sin\theta}{2\cos\theta}$$

for all real numbers $\theta$ satisfying $0 < \theta < \frac{1}{2}\pi$.




Multiplying by 2, and then taking reciprocals, we find that

$$\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{\cos\theta}{\sin\theta}$$

for all real numbers $\theta$ satisfying $0 < \theta < \frac{1}{2}\pi$. If we then multiply by $\sin\theta$, we obtain the inequalities

$$\cos\theta < \frac{\sin\theta}{\theta} < 1,$$

for all real numbers $\theta$ satisfying $0 < \theta < \frac{1}{2}\pi$.

Now, given any positive real number $\varepsilon$, there exists some positive real number $\delta$ satisfying $0 < \delta < \frac{1}{2}\pi$ such that $1 - \varepsilon < \cos\theta < 1$ whenever $0 < \theta < \delta$ (see Lemma 6.10). But then

$$1 - \varepsilon < \frac{\sin\theta}{\theta} < 1$$

whenever $0 < \theta < \delta$. These inequalities also hold when $-\delta < \theta < 0$, because the value of $\dfrac{\sin\theta}{\theta}$ is unchanged on replacing $\theta$ by $-\theta$. It follows that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, as required. ∎

**Corollary 6.12** *Let* $\cos\colon \mathbb{R} \to \mathbb{R}$ *be the cosine function whose value* $\cos\theta$, *for a given real number* $\theta$ *is the cosine of an angle of* $\theta$ *radians. Then*

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0.$$

**Proof** Basic trigonometrical identities ensure that

$$1-\cos\theta = 2\sin^2\tfrac{1}{2}\theta \quad \text{and} \quad \sin\theta = 2\sin\tfrac{1}{2}\theta\,\cos\tfrac{1}{2}\theta$$

for all real numbers $\theta$ (see Corollary 6.8 and Corollary 6.9). Therefore

$$\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta$$

for all real numbers $\theta$. It follows that

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = \lim_{\theta\to 0}\tan\tfrac{1}{2}\theta = 0,$$

and therefore

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} \times \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 0\times 1 = 0,$$

as required. ∎

**Proposition 6.13** *The derivatives of the sine and cosine functions satisfy*

$$\frac{d}{dx}(\sin x) = \cos x, \quad \textit{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

**Proof** Limits of sums, differences and products of functions are the corresponding sums, differences and products of the limits of those functions, provided that those limits exist (see Proposition 4.17). Also

$$\sin(x+h) = \sin x\,\cos h + \cos x\,\sin h$$

and

$$\cos(x+h) = \cos x\,\cos h - \sin x\,\sin h$$

for all real numbers $h$ (see Proposition 6.5). Applying these results, together with those of Proposition 6.11 and Corollary 6.12, we see that

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x\, \cos h + \cos x\, \sin h - \sin x}{h} \\ &= \cos x \lim_{h\to 0} \frac{\sin h}{h} - \sin x \lim_{h\to 0} \frac{1 - \cos h}{h} \\ &= \cos x. \end{aligned}$$

Similarly

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x\, \cos h - \sin x\, \sin h - \cos x}{h} \\ &= -\sin x \lim_{h\to 0} \frac{\sin h}{h} - \cos x \lim_{h\to 0} \frac{1 - \cos h}{h} \\ &= -\sin x, \end{aligned}$$

as required. ∎

**Corollary 6.14** *The derivative of the tangent function satisfies*

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x.$$

**Proof** Using the formulae for the derivatives of the sine and cosine functions (Proposition 6.13), together with the Quotient Rule for differentiation (Proposition 5.4) we find that

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{1}{\cos^2 x}\left(\frac{d}{dx}(\sin x)\cos x - \frac{d}{dx}(\cos x)\sin x\right) \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

as required. ∎

## 6.6 The Inverse Tangent Function

**Definition** The *inverse tangent function*

$$\arctan\colon \mathbb{R} \to (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$$

is defined so that, given any real number $x$, the quantity $\arctan x$ is the unique angle (specified in radian measure) for which $\tan(\arctan x)) = x$.

Thus the inverse tangent function is the unique function mapping the set $\mathbb{R}$ of real numbers into the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ with the property that $\tan(\arctan x)) = x$ for all real numbers $x$.

**Lemma 6.15** *The inverse tangent function*

$$\arctan\colon \mathbb{R} \to (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$$

*is continuous.*

**Proof** Let $s$ be a real number, and let $\beta = \arctan s$. Let some positive real number $\varepsilon$ be given. Then real numbers $\alpha$ and $\gamma$ can be chosen so that $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$, $-\frac{1}{2}\pi < \gamma < \frac{1}{2}\pi$, and

$$\beta - \varepsilon < \alpha < \beta < \gamma < \beta + \varepsilon.$$

Let $u = \tan\alpha$ and $v = \tan\gamma$. The tangent function is increasing. It follows that the inverse tangent function is also increasing. Thus if $x$ is a real number satisfying $u < x < v$ then $\alpha < \arctan x < \gamma$. Let $\delta$ be the smaller of the positive numbers $v - s$ and $s - u$. if $x$ is a real number satisfying $s - \delta < x < s + \delta$ then $u < x < v$. But then

$$\arctan s - \varepsilon < \alpha < \arctan x < \gamma < \arctan s + \varepsilon.$$

Thus the inverse tangent function arctan is continuous at $s$, as required. ∎

**Proposition 6.16** *The inverse tangent function* arctan *is differentiable, and*

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

*for all real numbers $x$.*

**Proof** Let $s$ be a real number. Then there exists a real number $\beta$ satisfying $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$ for which $\tan\beta = s$. Let $Q\colon (-\frac{1}{2}\pi, \frac{1}{2}\pi) \to \mathbb{R}$ be defined such that

$$Q(\theta) = \begin{cases} \dfrac{\tan\theta - \tan\beta}{\theta - \beta} & \text{if } \theta \neq \beta; \\ \dfrac{1}{\cos^2\beta} & \text{if } \theta = \beta. \end{cases}$$

Now the continuity of the tangent function, together with standard theorems on continuity, ensures that $Q(\theta)$ is a continuous function of $\theta$ when $\theta \neq \beta$. The function $Q$ is also continuous at $\beta$ because

$$\lim_{\theta\to\beta} Q(\theta) = \lim_{\theta\to\beta} \frac{\tan\theta - \tan\beta}{\theta - \beta} = \left.\frac{d}{d\theta}(\tan\theta)\right|_{\theta=\beta} = \frac{1}{\cos^2\beta} = Q(\beta)$$

(see Corollary 6.14). It follows that the function $Q(\theta)$ is continuous at $\theta$ for all values of $\theta$ satisfying $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

Now the inverse tangent function is continuous (Lemma 6.15), and compositions of continuous functions are continuous (Proposition 4.26). It follows that $Q(\arctan x)$ is a continuous function of $x$. Also $\tan\beta = s$. Therefore

$$\begin{aligned} \lim_{x\to s} Q(\arctan x) &= Q(\arctan s) = Q(\beta) = \frac{1}{\cos^2\beta} \\ &= 1 + \tan^2\beta = 1 + s^2. \end{aligned}$$

(see Proposition 6.4). But

$$\lim_{x\to s} Q(\arctan x) = \lim_{x\to s} \frac{x - s}{\arctan x - \arctan s}.$$

It follows that

$$\begin{aligned} \left.\frac{d}{dx}(\arctan x)\right|_{x=s} &= \lim_{x\to s} \frac{\arctan x - \arctan s}{x - s} \\ &= \lim_{x\to s} \frac{1}{Q(\arctan x)} = \frac{1}{1 + s^2}. \end{aligned}$$

Thus the inverse tangent function is differentiable, and moreover its derivative at any real number $x$ is equal to $\dfrac{1}{1+x^2}$, as required. ∎

## 6.7 The Inverse Sine and Cosine Functions

**Definition** The *inverse sine function*

$$\arcsin\colon [-1, 1] \to [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$$

is defined so that, given any real number $x$ satisfying $-1 \leq x \leq 1$, the quantity $\arcsin x$ is the unique angle (specified in radian measure) satisfying the inequalities

$$-\tfrac{1}{2}\pi \leq \arcsin x \leq \tfrac{1}{2}\pi$$

for which $\sin(\arcsin x)) = x$.

**Definition** The *inverse cosine function*

$$\arccos\colon [-1, 1] \to [0, \pi]$$

is defined so that, given any real number $x$ satisfying $-1 \leq x \leq 1$, the quantity $\arccos x$ is the unique angle (specified in radian measure) satisfying the inequalities

$$0 \leq \arcsin x \leq \pi$$

for which $\cos(\arccos x)) = x$.

The inverse sine and cosine functions are related by the identity

$$\arccos x = \frac{1}{2}\pi - \arcsin x$$

for all real numbers $x$ satisfying $-1 \leq x \leq 1$.

We explore the relationship between the inverse sine and inverse tangent functions.

**Lemma 6.17** *The inverse sine function* arcsin *is a differentiable function of* $x$ *on the interval* $(-1, 1)$ *which satisfies the identity*

$$\arcsin x = \arctan\left(\frac{x}{\sqrt{1 - x^2}}\right).$$

*when* $-1 < x < 1$.

**Proof** Let $x$ be a real number satisfying $-1 < x < 1$ and let $\theta$ be the unique real number in the range $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ that satisfies $\sin\theta = x$. Then $\cos\theta = \sqrt{1 - x^2}$, and therefore

$$\tan\theta = \frac{x}{\sqrt{1 - x^2}}.$$

It follows that

$$\arcsin x = \theta = \arctan\left(\frac{x}{\sqrt{1 - x^2}}\right).$$

Now it follows from the Chain Rule (Proposition 5.5) that any composition of differentiable functions is differentiable. Therefore the inverse sine function is differentiable at $x$ for all real numbers $x$ satisfying $-1 < x < 1$, as required. ∎

**Proposition 6.18** *The inverse sine function* arcsin *satisfies*

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

*for all real numbers* $x$ *satisfying* $-1 < x < 1$.

**Proof** Differentiating the identity $\sin(\arcsin x) = x$ with respect to $x$ using the Chain Rule (Proposition 5.5), we find that

$$\cos(\arcsin x)\frac{d}{dx}(\arcsin x) = 1.$$

Let $\theta = \arcsin x$. Then $x = \sin\theta$. It follows that

$$\cos(\arcsin x) = \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2},$$

and therefore

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}},$$

as required. ∎

**Remark** The inverse sine and tangent functions are related by the identity

$$\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right).$$

when $-1 < x < 1$. Differentiating the right hand side of this identity using the Chain Rule, and using the result that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

(Proposition 6.16), we find that

$$\begin{aligned}
&\frac{d}{dx}\left(\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right)\\
&\qquad = \frac{1}{1+\dfrac{x^2}{1-x^2}}\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right)\\
&\qquad = \frac{1-x^2}{(1-x^2)+x^2}\times\frac{(1-x^2)-\frac{1}{2}(-2x)x}{(1-x^2)^{\frac{3}{2}}}\\
&\qquad = (1-x^2)\times\frac{1}{(1-x^2)^{\frac{3}{2}}}\\
&\qquad = \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

This agrees with the formulae already found for the derivative of the inverse sine function (Proposition 6.18).

The inverse cosine function satisfies the identity

$$\arccos x = \frac{1}{2}\pi - \arcsin x.$$

It therefore follows from Proposition 6.18 that

$$\frac{d}{dx}(\arccos x) = -\frac{d}{dx}(\arcsin x) = -\frac{1}{\sqrt{1-x^2}}$$

for all real numbers $x$ satisfying $-1 < x < 1$.