Module MA1S11 (Calculus)
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Section 5: Differential Calculus

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Contents

5 Differential Calculus

5.1 Continuity of Differentiable Functions

Differentiable functions are continuous, as the following lemma shows.

Lemma 5.1 *Let s be some real number, and let f be a differentiable real-valued functions defined throughout some neighbourhood of s. Then the function f is continuous at s, and thus $\lim_{x \to s} f(x) = f(s)$.*

Proof The function f satisfies the identity

$$f(x) = \frac{f(x) - f(s)}{x - s} \times (x - s) + f(s)$$

for all real numbers x satisfying $x \neq s$ that lie sufficiently close to s. Now limits of sums and products of functions are the sums and products of the respective limits where those limits are defined (see Proposition 4.17). It follows that

$$\begin{aligned} \lim_{x \to s} f(x) &= \lim_{x \to s} \left(\frac{f(x) - f(s)}{x - s} \right) \times \lim_{x \to s} (x - s) + f(s) \\ &= f'(s) \times 0 + f(s) = f(s). \end{aligned}$$

This ensures that the function f is continuous at s. (Proposition 4.21). The result follows. ∎

5.2 Derivatives of Sums and Differences of Functions

Proposition 5.2 *Let s be some real number, and let f and g be real-valued functions defined throughout some neighbourhood of s. Suppose that the functions f and g are differentiable at s. Then $f + g$ and $f - g$ are differentiable at s, and*

$$(f + g)'(s) = f'(s) + g'(s), \qquad (f - g)'(s) = f'(s) - g'(s).$$

Proof Let x be a real number satisfying $x \neq s$ that is close enough to s to ensure that both $f(x)$ and $g(x)$ are defined at x. Now limits of sums and products of functions are the sums and products of the respective limits where those limits are defined (see Proposition 4.17). It follows that

$$\begin{aligned} &\lim_{x \to s} \frac{(f + g)(x) - (f + g)(s)}{x - s} \\ &\qquad = \lim_{x \to s} \frac{f(x) - f(s)}{x - s} + \lim_{x \to s} \frac{g(x) - g(s)}{x - s} \\ &\qquad = f'(s) + g'(s). \end{aligned}$$

Thus the function $f+g$ is differentiable at s, and $(f+g)'(s) = f'(s)+g'(s)$. An analogous proof shows that the function $f-g$ is also differentiable at s and $(f-g)'(s) = f'(s) - g'(s)$. ∎

5.3 The Product Rule

Proposition 5.3 (Product Rule) *Let s be some real number, and let f and g be differentiable real-valued functions defined throughout some neighbourhood of s. Let $f \cdot g$ denote the product function, defined so that $(f \cdot g)(x) = f(x)g(x)$ for all real numbers x for which both $f(x)$ and $g(x)$ are defined. Then the product function $f \cdot g$ is also differentiable at s, and*

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s).$$

Proof Let x be a real number satisfying $x \neq s$ that is close enough to s to ensure that both $f(x)$ and $g(x)$ are defined at x. Then

$$\begin{aligned} & \frac{f(x)g(x) - f(s)g(s)}{x-s} \\ &= \frac{f(x)-f(s)}{x-s} g(x) + f(s)\frac{g(x)-g(s)}{x-s}. \end{aligned}$$

Now $\lim_{x \to s} g(x) = g(s)$ because the differentiable function g is necessarily continuous at s (see Lemma 5.1). Also limits of sums and products of functions are the sums and products of the respective limits where those limits are defined (see Proposition 4.17). It follows that

$$\begin{aligned} & \lim_{x \to s} \frac{f(x)g(x) - f(s)g(s)}{x-s} \\ &= \lim_{x \to s} \frac{f(x)-f(s)}{x-s} \lim_{x \to s} g(x) + f(s) \lim_{x \to s} \frac{g(x)-g(s)}{x-s} \\ &= f'(s)g(s) + f(s)g'(s). \end{aligned}$$

Thus the function $f \cdot g$ is differentiable at s, and

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s),$$

as required. ∎

5.4 The Quotient Rule

Proposition 5.4 (Quotient Rule) *Let s be some real number, and let f and g be differentiable real-valued functions defined throughout some neighbourhood of s, where $g(s) \neq 0$. Let f/g denote the product function, defined* *so that $(f/g)(x) = f(x)/g(x)$ for all real numbers x for which $f(x)$ and $g(x)$ are defined and $g(x) \neq 0$. Then the quotient function f/g is differentiable at s, and*

$$(f/g)'(s) = \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2}.$$

Proof Let x be a real number satisfying $x \neq s$ that is close enough to s to ensure that both $f(x)$ and $g(x)$ are defined at x and that $g(x) \neq 0$. Then

$$\begin{aligned} \frac{f(x)}{g(x)} - \frac{f(s)}{g(s)} &= \frac{f(x)g(s) - f(s)g(x)}{g(x)g(s)} \\ &= \frac{(f(x) - f(s))\, g(s) - f(s)\, (g(x) - g(s))}{g(s)g(x)}. \end{aligned}$$

Now $\lim\limits_{x \to s} g(x) = g(s)$ because the differentiable function g is necessarily continuous at s (see Lemma 5.1). Also limits of sums, products and quotients of functions are the sums, products and quotients of the respective limits where those limits and quotients are defined (see Proposition 4.17). It follows that

$$\begin{aligned} &(f/g)'(s) \\ &= \lim_{x \to s} \frac{1}{x - s} \left(\frac{f(x)}{g(x)} - \frac{f(s)}{g(s)} \right) \\ &= \lim_{x \to s} \left(\frac{1}{g(x)g(s)} \right) \\ &\qquad \times \left(\lim_{x \to s} \frac{f(x) - f(s)}{x - s} g(s) - f(s) \lim_{x \to s} \frac{g(x) - g(s)}{x - s} \right) \\ &= \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2}, \end{aligned}$$

as required. ∎

5.5 The Chain Rule

Proposition 5.5 (Chain Rule) *Let s be some real number, let f be a real-valued function defined throughout some neighbourhood of s, and let g be a real-valued function defined throughout some neighbourhood of $f(s)$. Suppose that the function f is differentiable at s, and the function g is differentiable at $f(s)$. Then the composition function $g \circ f$ is differentiable at s, and*

$$(g \circ f)'(s) = g'(f(s))f'(s).$$

Proof Let $r = f(s)$, and let

$$Q(y) = \begin{cases} \dfrac{g(y) - g(r)}{y - r} & \text{if } y \neq r; \\ g'(r) & \text{if } y = r. \end{cases}$$

for values of y around r. By considering separately the cases when $f(x) \neq f(s)$ and $f(x) = f(s)$, we see that

$$g(f(x)) - g(f(s)) = Q(f(x))(f(x) - f(s)).$$

Now the function Q is continuous at r, where $r = f(s)$, because

$$\lim_{y \to r} Q(r) = \lim_{y \to r} \frac{g(y) - g(r)}{y - r} = g'(r) = Q(r)$$

(see Proposition 4.21). Also the function f is continuous at s, because it is differentiable at s (see Lemma 5.1). It follows that the composition function $Q \circ f$ is continuous at s (Proposition 4.26), and thus

$$\lim_{x \to s} Q(f(x)) = Q(f(s)) = g'(f(s))$$

(Proposition 4.21).

The limit of a product of functions is the product of the respective limits (see Proposition 4.17). Applying this result, we see that

$$\begin{aligned} (g \circ f)'(s) &= \lim_{x \to s} \frac{g(f(x)) - g(f(s))}{x - s} \\ &= \lim_{x \to s} Q(f(x)) \lim_{x \to s} \frac{f(x) - f(s)}{x - s} \\ &= g'(f(s))f'(s). \end{aligned}$$

The result follows. ∎

5.6 Rules for Differentiation

We summarize the basic rules for differentiation, expressed in the traditional language of real variables.

We regard a *real variable* as a real number x whose value can vary over some set D that is a subset of the set of real numbers. We say that a real variable y is a *dependent variable*, that can be represented as a function of a real variable x, where x takes values in a subset D of the set of real numbers, if the dependence of y of x can be represented by an equation of the form

$y = f(x)$, where $f\colon D \to \mathbb{R}$ is a real-valued function on the set D. We say that the dependent variable y is *differentiable* with respect to x if the function f that determines the dependence of y on x is a differentiable function. The derivative $\dfrac{dy}{dx}$ of y with respect to x is then the function whose value is equal to the derivative $f'(s)$ of the function f at $x = s$.

Proposition 5.6 *Let x be a real variable, taking values in a subset D of the real numbers, and let y, u and v dependent variables, expressible as functions of the independent variable x, that are differentiable with respect to x. Then the following results are valid:—*

(i) if $y = c$, where c is a real constant, then $\dfrac{dy}{dx} = 0$;

(ii) if $y = cu$, where c is a real constant, then $\dfrac{dy}{dx} = c\dfrac{du}{dx}$;

(iii) if $y = u + v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$;

(iv) if $y = x^q$, where q is a rational number, then $\dfrac{dy}{dx} = qx^{q-1}$;

(v) (Product Rule) *if $y = uv$ then* $\dfrac{dy}{dx} = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$;

(vi) (Quotient Rule) *if $y = \dfrac{u}{v}$ then* $\dfrac{dy}{dx} = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$;

(vii) (Chain Rule) *if y is expressible as a differentiable function of u, where u in turn is expressible as a differentiable function of x, then* $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$.

Proof Properties (i), (ii), (iii) follow directly from the definition of the derivative as a limit and from standard results concerning sums and products of limits (see Proposition 4.17). Property (v) is a restatement of Proposition 5.3. Property (vi) is a restatement of Proposition 5.4 Property (vii) is a restatement of Proposition 5.5. ∎

5.7 Local Maxima and Minima of Differentiable Functions

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number. We say that s belongs to the *interior* of D if there exist real numbers u and v satisfying $u < s < v$ such that the set D contains all real numbers x satisfying $u < x < v$.

We recall that, given real numbers u and v satisfying $u < v$, the interval (u, v) is defined so that

$$(u, v) = \{x \in \mathbb{R} \mid u < x < b\}.$$

Every real number s belonging to the interval (u, v) is then in the interior of (u, v). And a real number s is in the interior of a subset D of the set $\mathbb{R}$ of real numbers if and only if there exist real numbers u and v for which $u < s < v$ and $(u, v) \subset D$.

Remark It may be helpful to contemplate the definition of the interior of a set D of real numbers as follows: a real number s belonging to D is in the interior of D if and only if if it is completely surrounded by real numbers belonging to D. The formal definition merely makes precise what is meant by saying that s is "completely surrounded" by real numbers belonging to D.

For example, consider the (important) case in which $D = [a, b]$, where a and b are real numbers satisfying $a < b$ and

$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}.$$

The endpoints a and b of this interval are not completely surrounded by points of the interval. But those real numbers s that satisfy $a < s < b$ *are* completely surrounded by points of the interval $[a, b]$, and they belong to the interior of $[a, b]$, where that interior is defined in accordance with the formal definition given above.

Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers. The function f has a local minimum at s, where $s \in D$, if and only if there exists some positive real number δ such that $f(x) \ge f(s)$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$. Similarly the function f has a local maximum at s, where $s \in D$, if and only if there exists some positive real number δ such that $f(x) \le f(s)$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$.

(These definitions are to be found in Subsection 3.3.8 of the course notes.)

Proposition 5.7 *Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. Suppose that the function f has a local maximum or a local minimum at s, and that the function f is differentiable at s. Then $f'(s) = 0$.*

Proof Suppose that the function f attains a local minimum at s, where the real number s belongs to the interior of the set D. Suppose also that the function f is differentiable at s with derivative $f'(s)$. Then

$$f'(s) = \lim_{x \to s} \frac{f(x) - f(s)}{x - s} = \lim_{x \to s^+} \frac{f(x) - f(s)}{x - s}.$$

for all real numbers x greater than s that lie sufficiently close to s. But $f(x) \geq f(s)$ for all real numbers x that lie sufficiently close to s. It follows that

$$\frac{f(x) - f(s)}{x - s} \geq 0$$

for all real numbers x satisfying $x > s$ that lie sufficiently close to s. It follows that

$$\lim_{x \to s^+} \frac{f(x) - f(s)}{x - s} \geq 0$$

(see Proposition 4.18). It follows that $f'(s) \geq 0$.

Similarly

$$\frac{f(x) - f(s)}{x - s} \leq 0$$

for all real numbers x satisfying $x < s$ that lie sufficiently close to s. It follows that

$$f'(s) = \lim_{x \to s} \frac{f(x) - f(s)}{x - s} = \lim_{x \to s^-} \frac{f(x) - f(s)}{x - s} \leq 0.$$

Thus $f'(s) \geq 0$ and $f'(s) \leq 0$, and therefore $f'(s) = 0$.

Next suppose that the function f attains a local maximum at s, where s belongs to the interior of D and the function f is differentiable at s. Then the function $-f$ attains a local minumum at s, and therefore the derivative $-f'(s)$ of the function $-f$ at s is equal to zero. Thus $f'(s) = 0$. This completes the proof. ∎

Example Let

$$f(x) = 20x^{\frac{9}{4}} - 288x^{\frac{5}{4}} + 2700x^{\frac{1}{4}}.$$

for all real numbers x belonging to the interval $[1, 6]$, where

$$[1, 6] = \{x \in \mathbb{R} \mid 1 \leq x \leq 6\}.$$

Differentiating, we find that

$$f'(x) = 45x^{\frac{5}{4}} - 360x^{\frac{1}{4}} + 675x^{-\frac{3}{4}}$$

for all real numbers x belonging to the interval $[1, 6]$. Now

$$f'(x) = 45x^{-\frac{3}{4}}(x^2 - 8x + 15)$$

for all $x \in [1, 6]$. The derivative $f'(x)$ must be zero at any local maxima or minima in the interior of the interval $[1, 6]$. Now if $1 \leq x \leq 6$, and if $f'(x) = 0$, then either $x = 3$ or else $x = 5$, because 3 and 5 are the roots of the quadratic polynomial $x^2 - 8x + 15$.

Moreover the behaviour of this quadratic polynomial shows that $f'(x) > 0$ when $1 \leq x < 3$ and when $5 < x \leq 6$, and $f'(x) < 0$ when $3 < x < 5$. It follows that the function f is increasing on the intervals $[1, 3]$ and $[5, 6]$, but is decreasing on the interval $[3, 5]$. It follows that the function f attains a local maximum when $x = 3$, and attains a local minimum when $x = 5$. Calculating the values of $f(x)$ when x takes the values 1, 3, 5 and 6, we find that

$$f(1) = 2342, \quad f(3) = 2653.2052\ldots,$$

$$f(5) = 2631.8139\ldots, \quad f(6) = 2648.1231$$

to four decimal places. Applying the Intermediate Value Theorem (Theorem 4.28), we see that $f(x)$ takes on all real values between $f(1)$ and $f(3)$ as x increases from 1 to 3. It follows from the above calculuations that the range of the function is the interval $[f(1), f(3)]$, where $f(1) = 2342$ and $f(3) = 2653.2052$ to four decimal places.

5.8 Rolle's Theorem

Let $f\colon [a, b] \to \mathbb{R}$ be a continuous real-valued function defined on a closed interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$ and

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

It then follows from the Extreme Value Theorem (Theorem 4.29) that there exist real numbers u and v in the interval $[a, b]$ such that

$$f(u) \leq f(x) \leq f(v)$$

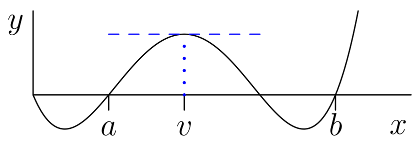
for all real numbers x belonging to the interval $[a, b]$. The Extreme Value Theorem was stated *without proof* earlier in the course.

We now apply the Extreme Value Theorem, together with result that derivatives of differentiable functions are zero at local maxima and minima in the interior of the domain of the function (Proposition 5.7) in order to prove *Rolle's Theorem*

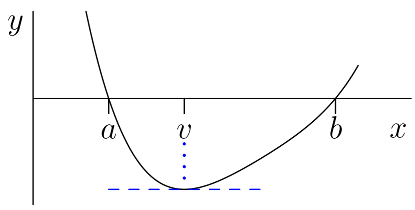
Theorem 5.8 (Rolle's Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.*

Proof The function f is continuous on the closed bounded interval $[a,b]$. It therefore follows from the Extreme Value Theorem that there must exist real numbers u and v in the interval $[a,b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$ (see Theorem 4.29).

Suppose that $f(v) > f(a)$. Then $f(v) > f(b)$, because $f(a) = f(b)$. It follows that $v \neq a$ and $v \neq b$. But $a \leq v \leq b$. It must therefore be the case that $a < v < b$. Moreover $f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$. The function f thus attains a local maximum at v, where v is in the interior of the interval $[a,b]$, and therefore $f'(v) = 0$ (see Proposition 5.7). In this case therefore we can take $s = v$.



Next suppose that $f(u) < f(a)$. Then $f(u) < f(b)$, because $f(a) = f(b)$. It follows that $u \neq a$ and $u \neq b$. But $a \leq u \leq b$. It must therefore be the case that $a < u < b$. Moreover $f(x) \leq f(u)$ for all real numbers x satisfying $a \leq x \leq b$. The function f thus attains a local minimum at u, where u is in the interior of the interval $[a,b]$, and therefore $f'(u) = 0$ (see Proposition 5.7). In this case therefore we can take $s = u$.



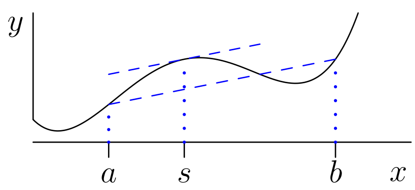
The only remaining case to consider is the case when both u and v are endpoints of the interval $[a, b]$. In that case the function f is constant on $[a, b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. ∎

5.9 The Mean Value Theorem

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 5.9 (The Mean Value Theorem) *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$f(b) - f(a) = f'(s)(b - a).$$



Proof Let $p\colon [a, b] \to \mathbb{R}$ be the function defined so that

$$p(x) = \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b) = mx + k,$$

where

$$m = \frac{f(b) - f(a)}{b - a} \quad \text{and} \quad k = \frac{bf(a) - af(b)}{b - a}$$

Then $p(a) = f(a)$, $p(b) = f(b)$ and $p'(x) = m$ for all real numbers x satisfying $a \leq x \leq b$. (The equation $y = p(x)$ is then the equation of the line segment that joins the points $(a, f(a))$ and $(b, f(b))$ on the graph of f at $x = a$ and $x = b$.)

Next let $g\colon [a, b] \to \mathbb{R}$ be the function defined such that $g(x) = f(x) - p(x)$ for all real numbers x satisfying $a \leq x \leq b$. Then $g(a) = g(b) = 0$, because $f(a) = p(a)$ and $f(b) = p(b)$, and $g'(x) = f'(x) - m$ for all real numbers x satisfying $a \leq x \leq b$. It follows from Rolle's Theorem (Theorem 5.8) that there exists some real number s satisfying $a < s < b$ for which $g'(s) = 0$. But then

$$f'(s) = g'(s) + m = m = \frac{f(b) - f(a)}{b - a},$$

and thus $f(b) - f(a) = f'(s)(b - a)$, as required. ∎

5.10 Twice-Differentiable Functions

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a real number in the interior of D. The function f is said to be *twice-differentiable* at s if the derivative f' is defined and differentiable around s. The *second derivative* $f''(s)$ of a twice-differentiable function f at s is the value of the derivative of the derivative of f at s.

Let x be a real variable that ranges over a subset D of the set of real numbers, and let the dependent variable y be defined so that $y = f(x)$ for all values of x that belong to D, where $f\colon D \to \mathbb{R}$ is a twice-differentiable function on D. The first derivative $\dfrac{dy}{dx}$ of y with respect to x then satisfies

$$\frac{dy}{dx} = f'(x)$$

throughout D, and the second derivative $\dfrac{d^2y}{dx^2}$ of y with respect to x satisfies

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$$

throughout D.

5.11 The Second Derivative Test for Local Minima and Maxima

Proposition 5.10 (Second Derivative Test for Local Minimum) *Let $f\colon D \to \mathbb{R}$ be a twice-differentiable real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. Suppose that $f'(s) = 0$ and $f''(s) > 0$. Then the function f has a local minimum at s.*

Proof The first derivative f' of f satisfies

$$\lim_{x \to s} \frac{f'(x) - f'(s)}{x - s} = f''(s) > 0.$$

It follows that there exists some positive real number δ such that $x \in D$ and

$$\frac{f'(x) - f'(s)}{x - s} > \tfrac{1}{2} f''(s) > 0$$

whenever $s - \delta < x < s + \delta$ and $x \neq s$. But $f'(s) = 0$. It follows that

$$\frac{f'(x)}{x - s} > 0$$

whenever $s - \delta < x < s + \delta$ and $x \neq s$, and therefore $f'(x) > 0$ whenever $s < x < s + \delta$, and $f'(x) < 0$ whenever $s - \delta < x < s$.

Now it follows from the Mean Value Theorem (Theorem 5.9) that if x is a real number satisfying $s < x < s + \delta$ then there exists some real number v satisfying $s < v < x$ for which $f(x) - f(s) = f'(v)(x - s)$. But the derivative $f'(v)$ of f at v must then satisfy $f'(v) > 0$. It follows that $f(x) > f(s)$ whenever $s < x < s + \delta$.

It also follows from the Mean Value Theorem (Theorem 5.9) that if x is a real number satisfying $s - \delta < x < s$ then there exists some real number u satisfying $x < u < s$ for which $f(s) - f(x) = f'(u)(s - x)$. But the derivative $f'(u)$ of f at u must then satisfy $f'(v) < 0$. It follows that $f(x) > f(s)$ whenever $s - \delta < x < s$. We conclude from these results that the function f attains a local minimum at s, as required. ∎

Corollary 5.11 (Second Derivative Test for Local Maximum) *Let $f: D \to \mathbb{R}$ be a twice-differentiable real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. Suppose that $f'(s) = 0$ and $f''(s) < 0$. Then the function f has a local maximum at s.*

Proof This result follows immediately on applying Proposition 5.10 to the function $-f$. ∎

Let $f: D \to \mathbb{R}$ be a twice-differentiable real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. Suppose that $f'(s) = 0$. If $f''(s) > 0$ then the function f has a local minimum at s. If $f''(s) < 0$ then the function f has a local maximum at s. But if $f''(s) = 0$ then one is not in a position to draw any conclusion about whether there is a local minimum or maximum at s.

Example Let $f: \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = x^4$ for all real numbers x. Then $f'(0) = 0$ and $f''(0)$. The function f has a local minimum at zero.

Example Let $g: \mathbb{R} \to \mathbb{R}$ be defined so that $g(x) = -x^4$ for all real numbers x. Then $g'(0) = 0$ and $g''(0)$. The function g has a local maximum at zero.

Example Let $h: \mathbb{R} \to \mathbb{R}$ be defined so that $h(x) = x^3$ for all real numbers x. Then $h'(0) = 0$ and $h''(0)$. The function h has neither a local minimum nor a local maximum at zero.

5.12 Concavity and Points of Inflection

Let $f: D \to \mathbb{R}$ be a twice-differentiable function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that $f''(x) > 0$ for all $x \in I$. If u and v are real numbers belonging to the interval I that satisfy $u < v$ then from the Mean Value Theorem (Theorem 5.9) that there exists some real number s satisfying $u < s < v$ for which $f'(v) - f'(u) = f''(s)(v - u)$. But then $s \in I$, and therefore $f''(s) > 0$. It follows that $f'(u) < f'(v)$ for all real numbers u and v in the interval I. The graph of the function f thus becomes ever steeper as x increases through the interval I.

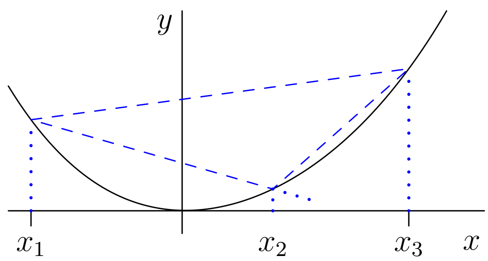
Now let x_1, x_2 and x_3 be real numbers belonging to the interval I that satisfy $x_1 < x_2 < x_3$. It follows from the Mean Value Theorem that there exist real numbers u and v satisfying $x_1 < u < x_2 < v < x_3$ such that

$$\frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(v) \quad \text{and} \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(u).$$

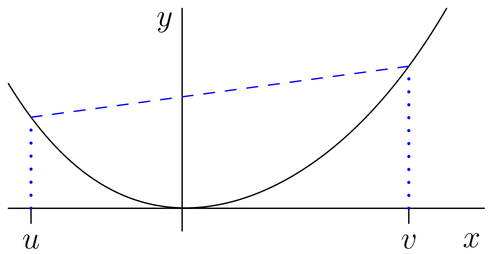
But $f'(u) < f'(v)$ because the second derivative of f is positive throughout the interval I. It follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

Thus the slope of the line segment joining the points $(x_2, f(x_2))$ and $(x_3, f(x_3))$ is greater than the slope of the line segment joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. It follows from this that the point $(x_3, f(x_3))$ lies above the line passing through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and therefore the point $(x_2, f(x_2))$ lies below the line joining the points $(x_1, f(x_1))$ and $(x_3, f(x_3))$. Moreover this argument applies for all values of x_2 that lie between x_1 and x_3. It follows that the graph of the function lies under the line segment joining the points $(x_1, f(x_1))$ and $(x_3, f(x_3))$.



Definition Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that, given real numbers u and v belonging to I that satisfy $u < v$, the line segment joining the point $(u, f(u))$ to the point $(v, f(v))$ lies above the graph of the function. Then the graph of the function is said to be *concave upwards* on the interval I.



The following result follows immediately from the preceding discussion.

Proposition 5.12 *Let $f: D \to \mathbb{R}$ be a twice-differentiable function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that $f''(x) > 0$ for all $x \in I$. Then the graph of the function is concave upwards on I.*

Definition Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that, given real numbers u and v belonging to I that satisfy $u < v$, the line segment joining the point $(u, f(u))$ to the point $(v, f(v))$ lies below the graph of the function. Then the graph of the function is said to be *concave downwards* on the interval I.

Corollary 5.13 *Let $f: D \to \mathbb{R}$ be a twice-differentiable function defined on a subset D of the set of real numbers, and let I be an interval satisfying $I \subset D$. Suppose that $f''(x) < 0$ for all $x \in I$. Then the graph of the function is concave downwards on I.*

Proof The result follows immediately on applying Proposition 5.12 to the function $-f$. ∎

Definition Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a real number belonging to the interior of D. The point $(s, f(s))$ is said to be a *point of inflexion* of the graph of the function if s is common endpoint of an interval where the graph of the function is concave upwards and an interval where the graph of the function is concave downwards

Proposition 5.14 *Let $f: D \to \mathbb{R}$ be a twice-differentiable function defined on a subset D of the set of real numbers, where the second derivative f'' is continuous on D, and let s be a point in the interior of D. Suppose that s determines a point of inflexion on the graph of the function f. Then $f''(s) = 0$.*

Proof If it were the case that $f''(s) > 0$ then the second derivative would be positive around s, and therefore the real number s would be in the interior of an interval on which the graph of the function is concave upwards (see Proposition 5.12). This is not possible. Therefore it cannot be the case that $f''(s) > 0$. An analogous argument shows that it cannot be the case that $f''(s) < 0$. (Indeed if the second derivative of f were negative at s then the second derivative of $-f$ would be positive at s, and we have shown that this is impossible.) Therefore $f''(s) = 0$, as required. ∎

5.13 The Newton-Raphson Method

Let $f: D \to \mathbb{R}$ be a differentiable function defined on a subset D of the set of real numbers. A *zero* (or *root*) of the function f is a real number x belonging to the domain of the function that satisfies the equation $f(x) = 0$.

Suppose we wish to locate zeros of the function f. There is an iterative method for locating zeros by successive approximations, generally known as the *Newton-Raphson Method*, which may in the appropriate circumstances determine the value of a zero of the function to a high degree of precision.

Let x_n be a real number in the domain D of the differentiable function $f: D \to \mathbb{R}$. Then the tangent line to the graph of the function f at $(x_n, f(x_n))$ satisfies the equation

$$y = f(x_n) + f'(x_n)(x - x_n),$$

where $f'(x_n)$ denotes the derivative of the function f at x_n. This tangent line crosses the x-axis at the point $(x_{n+1}, 0)$, where

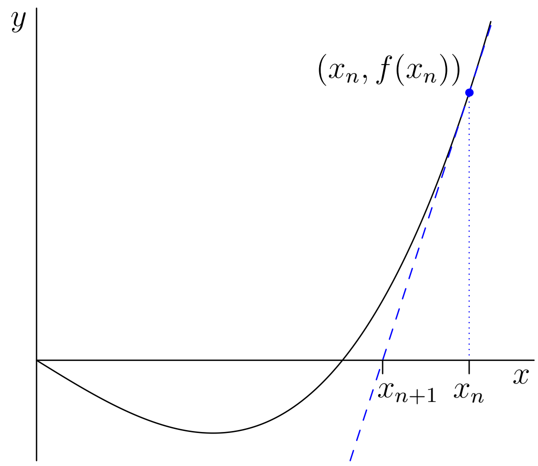
$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n).$$

Solving this equation for x_{n+1}, we find that

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}.$$

It follows that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$



The *Newton-Raphson* method for locating zeros of a differentiable function involves choosing an approximation x_1 to the zero, and then computing the sequence $x_1, x_2, x_3, x_4, \ldots$ of successive approximations to the zero so that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

for all positive integers n.

Example Let $f(x) = x^3 - 2x$ for all real numbers x. Then $f'(x) = 3x^2 - 2$. We take $x_1 = 2$ as our initial approximation to a root of $f(x)$. Successive approximations are then determined by the Newton-Raphson method, so that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^3 - 2x}{3x^2 - 2}$$

for all natural numbers n. A computer-assisted calculation yields the following values for the successive approximations obtained:—

$$\begin{aligned}
x_1 &= 2.0, \\
x_2 &= 1.6, \\
x_3 &= 1.44225352112676 06\ldots, \\
x_4 &= 1.415010636743953\ldots, \\
x_5 &= 1.4142142353546963\ldots, \\
x_6 &= 1.4142135623735754\ldots, \\
x_7 &= 1.4142135623730951\ldots, \\
x_8 &= 1.4142135623730951\ldots,
\end{aligned}$$