Module MA1S11 (Calculus)
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Section 4: Limits And Derivatives

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Contents

4 Limits and Derivatives of Functions of a Real Variable

4.1 Secant Lines and Tangent Lines

Proposition 4.1 *Let s, t, p and q be real numbers, where $s \neq t$. Then the line passing through the points (s, p) and (t, q) is determined by the equation*

$$y = \frac{t - x}{t - s}\, p + \frac{x - s}{t - s}\, q.$$

The gradient of this line is

$$= \frac{q - p}{t - s},$$

and line intersects the coordinate axes at the points

$$\left(\frac{sq - tp}{q - p}, 0 \right) \quad \textit{and} \quad \left(0, \frac{tp - sq}{t - s} \right).$$

Proof The equation in the statement of the proposition relates the real variables x and y through an equation of the form $y = mx + k$ for appropriate constants m and k. Equations of this form determine lines. Now

$$\frac{t - x}{t - s} = 1 \text{ and } \frac{x - s}{t - s} = 0 \text{ when } x = s.$$

Similarly

$$\frac{t - x}{t - s} = 0 \text{ and } \frac{x - s}{t - s} = 1 \text{ when } x = t.$$

It follows that if

$$y = \frac{t - x}{t - s}\, p + \frac{x - s}{t - s}\, q$$

then $y = p$ when $x = s$, and $y = q$ when $x = t$. Thus this line passes through the points (s, p) and (t, q).

The gradient of the line satisfies the given formula since this quantity is the ratio of the increments of y and x between $x = s$ and $x = t$. The line intersects the x-axis when $y = 0$, and x then satisfies the equation

$$(t - x)p + (x - s)q = 0.$$

It follows that this line intersects the x-axis at the point

$$\left(\frac{sq - tp}{q - p}, 0 \right).$$

On setting $x = 0$ we find that the line intersects that y-axis at $(0, y_0)$, where

$$\left(0, \frac{tp - sq}{t - s}\right).$$

This completes the proof. ∎

Example Let a and c be real constants satisfying $a > 0$ and $c > 0$, and let $f\colon (0, +\infty) \to \mathbb{R}$ be the function on the positive real numbers defined such that

$$f(x) = ax + \frac{c}{x}$$

for all positive real numbers x. Let s be a positive real constant, and let t vary over positive real numbers distinct from s. In the context of this example, a *secant line* is a line that cuts the curve $y = f(x)$ in two points: a *tangent line* is a line which "touches" the curve in a single point.

The secant line cutting the curve $y = f(x)$ at the points $(s, f(s))$ and $(t, f(t))$ is determined by an equation of the form

$$y = m_s(t)x + k_s(t),$$

where

$$k_s\colon (0, +\infty) \setminus \{s\} \to \mathbb{R} \quad \text{and} \quad m_s\colon (0, +\infty) \setminus \{s\} \to \mathbb{R}$$

are the functions defined over the set $(0, +\infty) \setminus \{s\}$ of positive real numbers distinct from s.

Now the secant line cutting the curve $y = f(x)$ at the points $(s, f(s))$ and $(t, f(t))$ has equation

$$y = m_s(t)x + k_s(t),$$

Moreover we can apply Proposition 4.1 in order to obtain expressions for $m_s(t)$ and $k_s(t)$ in terms of s and t.

Let $p = f(s)$ and $q = f(t)$. Then

$$p = as + \frac{c}{s} \quad \text{and} \quad q = at + \frac{c}{t}.$$

The gradient $m_s(t)$ of the line through the points (s, p) and (t, q) then satisfies the equation

$$m_s(t) = \frac{q - p}{t - s}.$$

Now

$$\begin{aligned} q - p &= at + \frac{c}{t} - as - \frac{c}{s} = a(t - s) + \frac{c}{st}(s - t) \\ &= (t - s)\left(a - \frac{c}{st}\right). \end{aligned}$$

It follows that

$$m_s(t) = a - \frac{c}{st} \quad (\text{where } s > 0,\ \ t > 0 \ \text{ and } \ s \neq t).$$

It also follows from Proposition 4.1 that

$$k_s(t) = \frac{tp - sq}{t - s},$$

where, for the function we are considering here,

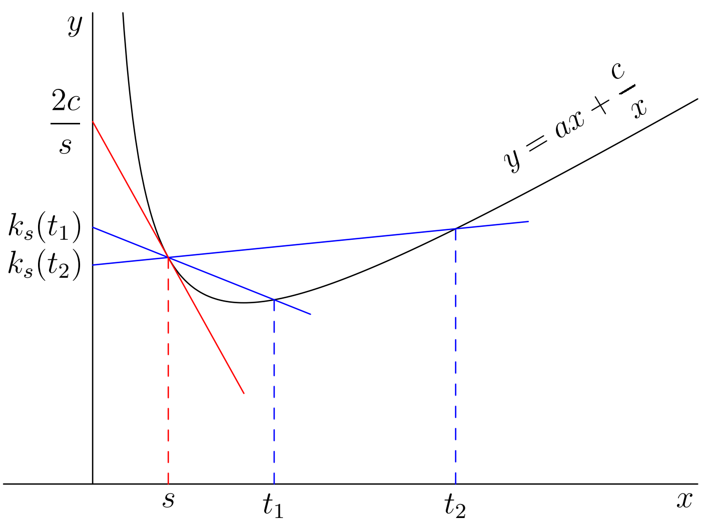
$$p = as + \frac{c}{s} \quad \text{and} \quad q = at + \frac{c}{t}.$$

Now

$$\begin{aligned} tp - sq &= \frac{ct}{s} - \frac{cs}{t} = \frac{ct^2}{st} - \frac{cs^2}{st} \\ &= \frac{c(t^2 - s^2)}{st} = (t - s)\,\frac{c(t + s)}{st} \\ &= (t - s)\left(\frac{c}{s} + \frac{c}{t}\right). \end{aligned}$$

It follows that

$$k_s(t) = \frac{c}{s} + \frac{c}{t} \quad (\text{where } s > 0,\ \ t > 0 \ \text{ and } \ s \neq t).$$



We now summarize the results obtained so far. We are investigating the secant line to the graph of the function $f\colon (0, +\infty) \to \mathbb{R}$, where this function is determined by positive constants a and c so that

$$f(x) = ax + \frac{c}{x}.$$

We regard the positive real number s as fixed, and consider the coefficients of the secant line that intersects the curve $y = f(x)$ at $x = s$ and $x = t$, where $t > 0$ and $t \neq s$.

The results verified above show that this secant line is of the form

$$y = m_s(t)x + k_s(t),$$

where

$$k_s\colon (0, +\infty) \setminus \{s\} \to \mathbb{R} \quad \text{and} \quad m_s\colon (0, +\infty) \setminus \{s\} \to \mathbb{R}$$

are the functions defined over the set $(0, +\infty) \setminus \{s\}$ of positive real numbers distinct from s so that

$$m_s(t) = a - \frac{c}{st}$$

and

$$k_s(t) = \frac{c(s+t)}{st} = \frac{c}{s} + \frac{c}{t}.$$

for all positive real numbers t distinct from s.

Inspecting these equations, we can draw the following conclusions:—

- $m_s(t)$ increases as t increases;
- $k_s(t)$ decreases as t increases;
- $m_s(t) < a - \dfrac{c}{s^2}$ if $0 < t < s$;
- $m_s(t) > a - \dfrac{c}{s^2}$ if $t > s$;
- $k_s(t) > \dfrac{2c}{s}$ if $0 < t < s$;
- $k_s(t) < \dfrac{2c}{s}$ if $t > s$;
- there is no secant line passing through s with gradient $a - \dfrac{c}{s^2}$, and therefore the line passing through s with this gradient is not a secant line;
- $m_s(t)$ "approaches" $a - \dfrac{c}{s^2}$ as t "approaches" s;
- $k_s(t)$ "approaches" $\dfrac{2c}{s}$ as t "approaches" s.

However it is necessary in mathematics to have a precise criterion to determine whether or not variable quantities "approach" fixed values. The criterion is expressed in various ways depending on context, but for variable quantities such as we are dealing with here, we say that some quantity expressible as a function $F(x)$ of a real variable x "tends to" a *limit* L as x "tends to" a fixed quantity s if, given any strictly positive real number ε, there exists a strictly positive real number δ such that

$$L - \varepsilon < F(x) < L + \varepsilon$$

for all real numbers x in the domain of the function F that satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

Accordingly we investigate whether or not the fixed quantity $\dfrac{2c}{s}$ is the limit of $k_s(t)$ as t tends to s in accordance with this precise definition of limits.

One way of approaching this is to determine what values of t are mapped to $\dfrac{2c}{s}+\varepsilon$ and $\dfrac{2c}{s}-\varepsilon$ by the function k_s, where ε is some given strictly positive real number. Now

$$\begin{aligned} & k_s(t) = \frac{2c}{s} + \varepsilon \\ \iff\quad & \frac{c}{s} + \frac{c}{t} = \frac{2c}{s} + \varepsilon \\ \iff\quad & \frac{c}{t} = \frac{c}{s} + \varepsilon \\ \iff\quad & \frac{cs}{t} = c + s\varepsilon \\ \iff\quad & t = \frac{cs}{c + s\varepsilon} \end{aligned}$$

Similarly

$$\begin{aligned} & k_s(t) = \frac{2c}{s} - \varepsilon \\ \iff\quad & \frac{c}{s} + \frac{c}{t} = \frac{2c}{s} - \varepsilon \\ \iff\quad & \frac{c}{t} = \frac{c}{s} - \varepsilon \\ \iff\quad & \frac{cs}{t} = c - s\varepsilon \\ \iff\quad & t = \frac{cs}{c - s\varepsilon} \end{aligned}$$

Now choose a positive real number δ small enough to ensure that both

$$0 < \delta \leq s - \frac{cs}{c + s\varepsilon}$$

and

$$0 < \delta \leq \frac{cs}{c - s\varepsilon} - s,$$

which we can do, for example, by taking δ to be the minimum of the quantities

$$s - \frac{cs}{c + s\varepsilon} \quad \text{and} \quad \frac{cs}{c - s\varepsilon} - s.$$

Then

$$\frac{cs}{c + s\varepsilon} \leq s - \delta < s + \delta \leq \frac{cs}{c - s\varepsilon}$$

We now determine which of the two quantities

$$s - \frac{cs}{c + s\varepsilon} \quad \text{and} \quad \frac{cs}{c - s\varepsilon} - s$$

is the smaller. Now

$$s - \frac{cs}{c + s\varepsilon} = \frac{s(c + s\varepsilon) - cs}{c + s\varepsilon} = \frac{s^2\varepsilon}{c + s\varepsilon}$$

and

$$\frac{cs}{c - s\varepsilon} - s = \frac{cs - s(c - s\varepsilon)}{c - s\varepsilon} = \frac{s^2\varepsilon}{c - s\varepsilon}.$$

Now c, s and ε are all positive, and therefore

$$\frac{s^2\varepsilon}{c + s\varepsilon} < \frac{s^2\varepsilon}{c - s\varepsilon}.$$

It follows that we should take

$$\delta = \frac{s^2\varepsilon}{c + s\varepsilon}.$$

It follows that if a real number t satisfies both

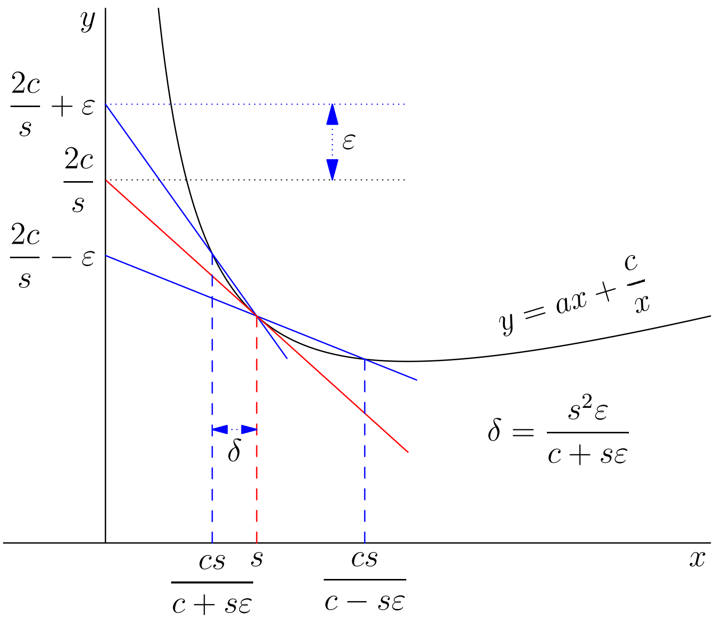
$$s - \delta < t < s + \delta \quad \text{and} \quad t \neq s,$$

where

$$0 < \delta \leq \frac{s^2\varepsilon}{c + s\varepsilon},$$

then

$$\frac{cs}{c + s\varepsilon} < t < \frac{cs}{c - s\varepsilon},$$



and therefore

$$\frac{2c}{s} - \varepsilon < k_s(t) < \frac{2c}{s} + \varepsilon.$$

Thus, given any positive real number ε, there always exists the corresponding positive real number δ that is small enough to ensure that $k_s(t)$ is within a distance ε of its limiting value $\frac{2c}{s}$ for all real numbers t distinct from s that are nevertheless within a distance δ of the fixed value s.

We have therefore proved, formally, that $\frac{2c}{s}$ is the limit of $k_s(t)$ as t tends to s. This conclusion can be recorded by writing

$$\lim_{t\to s} k_s(t) = \frac{2c}{s}.$$

We consider also whether or not, according to the formal definition of limits described above, it is the case that

$$\lim_{t\to s} m_s(t) = a - \frac{c}{s^2}.$$

Thus we need to determine whether it is the case that, given any positive real number ε, a positive real number δ exists so that

$$a - \frac{c}{s^2} - \varepsilon < m_s(t) < a - \frac{c}{s^2} + \varepsilon$$

for all real numbers t satisfying both

$$s - \delta < t < s + \delta \quad \text{and} \quad t \neq s.$$

It saves effort to note that, for $t \neq s$, the values $k_s(t)$ and $m_s(t)$ of the functions k_s and m_s at t are related by the equation

$$m_s(t) = a + \frac{c}{s^2} - \frac{1}{s}\,k_s(t).$$

Indeed

$$m_s(t) = a - \frac{c}{st} \quad \text{and} \quad k_s(t) = \frac{c}{s} + \frac{c}{t},$$

and therefore

$$m_s(t) + \frac{1}{s}\,k_s(t) = a + \frac{c}{s^2}.$$

Thus if

$$\frac{2c}{s} - s\varepsilon < k_s(t) < \frac{2c}{s} - s\varepsilon$$

then

$$a - \frac{c}{s^2} - \varepsilon < m_s(t) < a - \frac{c}{s^2} + \varepsilon.$$

It follows from what has already been achieved that the necessary inequalities will be satisfied provided that both

$$s - \delta < t < s + \delta \quad \text{and} \quad t \neq s,$$

where

$$0 < \delta \leq \frac{s^3\varepsilon}{c + s^2\varepsilon}.$$

Thus it is indeed the case that

$$\lim_{t \to s} m_s(t) = a - \frac{c}{s^2}.$$

Now, for fixed s, $m_s(t)$ is the gradient of the secant line passing through the points $(s, f(s))$ and $(t, f(t))$, where

$$f(x) = ax + \frac{c}{x}$$

for all real numbers s. It follows that

$$m_s(t) = \frac{f(t) - f(s)}{t - s}$$

for all positive real numbers t distinct from s. We conclude therefore that

$$\lim_{t \to s} \frac{f(t) - f(s)}{t - s} = a - \frac{c}{s^2}.$$

The limit appearing on the left hand side of this equation is the *derivative* of the function f at s. It measures the "instantaneous" rate of change of $f(x)$ with respect to x at $x = s$.

Now the "dependent" real variable y is related to the "independent" real variable x through the equation $y = f(x)$, which expresses y as a function of x. To determine the rate of change of y with respect to x when $x = s$, the increment $t - s$ in the variable x as x increases from s to t is traditionally denoted by Δx, and similarly the corresponding increment $f(t) - f(s)$ is traditionally denoted by Δy. The quantity

$$\frac{f(t) - f(s)}{t - s}$$

that measures the gradient of the secant line intersecting the graph of the function f at $(s, f(s))$ and $(t, f(t))$ is traditionally referred to as the *difference quotient* determined by the given increment, and we see that

$$\frac{f(t) - f(s)}{t - s} = \frac{\Delta y}{\Delta x}.$$

Thus the difference quotient is the ratio of the corresponding increments in the variables x and y as x increases from s to $s + \Delta x$. The relevant definitions and results therefore ensure that the *derivative* $f'(s)$ of the function f at $x = s$ thus satisfies

$$\begin{aligned} f'(s) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(s + \Delta x) - f(s)}{\Delta x} \\ &= \lim_{t \to s} \frac{f(t) - f(s)}{t - s} = a - \frac{c}{s^2}. \end{aligned}$$

This example has in fact included both a discussion of the formal definition of limits, and also an exercise in "differentiation from first principles", which involves determining the derivative of a function from the basic definitions, possibly also making use of basic results following from those basic definitions.

4.2 Limits of Functions of a Real Variable

Let s be a real number, and let D be a subset of the set $\mathbb{R}$ of real numbers. The set D is said to be a *neighbourhood* of s if there exists some positive real number δ such that $(s - \delta, s + \delta) \subset D$, where

$$(s - \delta, s + \delta) = \{x \in \mathbb{R} \mid s - \delta < x < s + \delta).$$

The definition of neighbourhoods ensures that if s is a real number, if $f: D \to \mathbb{R}$ is a real-valued function defined on a subset D of $\mathbb{R}$, and if $D \cup \{s\}$ is a neighbourhood of s then there exists some strictly positive real number δ_0 such that $f(x)$ is defined for all real numbers x satisfying both

$$s - \delta_0 < x < s + \delta_0 \quad \text{and} \quad x \neq s.$$

Definition Let s and L be real numbers, and let $f: D \to \mathbb{R}$ be a real-valued function defined over a subset D of $\mathbb{R}$ for which $D \cup \{s\}$ is a neighbourhood of s. We say that L is the *limit* of $f(x)$ as x tends to s, and write

$$\lim_{x \to s} f(x) = L,$$

if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $L - \varepsilon < f(x) < L + \varepsilon$ for all real numbers x in D that satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

Example We show that

$$\lim_{x \to 1} \frac{3x^2 - 4x + 1}{x - 1} = 2.$$

Indeed let $f: \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be defined such that

$$f(x) = \frac{3x^2 - 4x + 1}{x - 1}$$

for all real numbers x satisfying $x \neq 1$. Now $3x^2 - 4x + 1 = (3x - 1)(x - 1)$ for all real numbers x. It follows that $f(x) = 3x - 1$ whenever $x \neq 1$. Given any strictly positive real number ε, let $\delta = \frac{1}{3}\varepsilon$. If $x \neq 1$ and $1 - \delta < x < 1 + \delta$ then $2 - \varepsilon < f(x) < 2 + \varepsilon$, and thus the definition of limits is satisfied.

Example Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined such that

$$f(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then the limit of $f(x)$ as x tends to 0 does not exist.

Indeed suppose that this limit were to exist, and suppose that it were equal to the real number L. Then (applying the "epsilon-delta" definition of limits with $\varepsilon = \frac{1}{4}$ there would exist some strictly positive real number δ with the property that

$$L - \tfrac{1}{4} < f(x) < L + \tfrac{1}{4}$$

for all non-zero real numbers x satisfying $-\delta < x < \delta$. It would then follow that

$$-\tfrac{1}{2} \le f(u) - f(v) \le \tfrac{1}{2}$$

for all non-zero real numbers u and v satisfying $-\delta < u < \delta$ and $-\delta < v < \delta$.

But, given such a real number δ let $u = -\frac{1}{2}\delta$ and $v = \frac{1}{2}\delta$. Then $f(u) = 0$ and $f(v) = 1$, and therefore $f(v) - f(u) = 1$. It follows that no strictly positive real number δ could exist with the stated properties, and thus the definition of limits cannot be satisfied by the function f at zero.

The example just discussed exemplifies the phenomenon that, for a function $f: D \to \mathbb{R}$ defined over a subset D of $\mathbb{R}$, the limit $\lim_{x \to s} f(x)$ of $f(x)$ as x tends to some fixed value s will not exist if the function has a "jump" at s.

Example Consider the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined such that

$$f(x) = \sin\left(\frac{\pi}{2x}\right)$$

for all non-zero real numbers x. First we review the values of $\sin(\frac{1}{2}\pi j)$ when j is an integer. These values are determined as follows:

$$\sin(\tfrac{1}{2}\pi j) = \begin{cases} 0 & \text{if } j \text{ is an even integer;} \\ 1 & \text{if } j - 1 \text{ is divisible by 4;} \\ -1 & \text{if } j - 3 \text{ is divisible by 4.} \end{cases}$$

It follows that

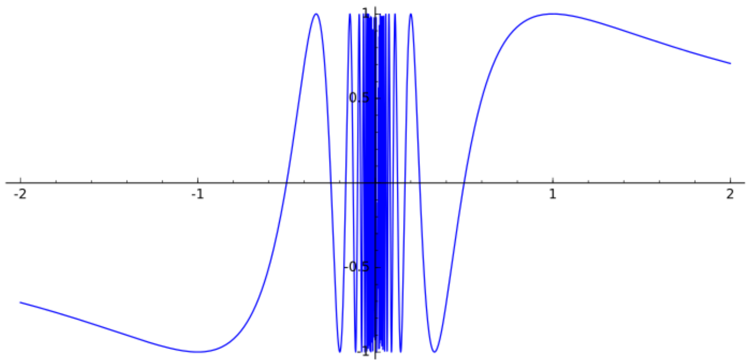
$$f(x) = \sin\left(\frac{\pi}{2x}\right) = \begin{cases} 0 & \text{if } x = \dfrac{1}{2k} \text{ for some non-zero integer } k; \\ 1 & \text{if } x = \dfrac{1}{4k+1} \text{ for some integer } k; \\ -1 & \text{if } x = \dfrac{1}{4k+3} \text{ for some integer } k. \end{cases}$$

The function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ does not tend to any limit as x tends to 0. Indeed suppose that this function were to converge to some limit L. Taking $\varepsilon = \frac{1}{4}$ in the "epsilon-delta" criterion that must be satisfied for a limit to exist, we see that there would exist some positive real number δ such that

$$L - \tfrac{1}{4} < f(x) < L + \tfrac{1}{4}$$

for all non-zero real numbers x satisfying $-\delta < x < \delta$. It would then follow that

$$-\tfrac{1}{2} < f(u) - f(v) < \tfrac{1}{2}$$



Graph of $\sin\left(\dfrac{\pi}{2x}\right)$ as a function of x for $x \neq 0$.

(Plot generated using SageMath.)

for all non-zero real numbers u and v satisfing $-\delta < u < \delta$ and $-\delta < v < \delta$. But a positive integer k could be chosen large enough to ensure that

$$\frac{1}{4k+1} < \delta.$$

Letting

$$u = \frac{1}{4k+1} \quad \text{and} \quad v = \frac{1}{4k+3},$$

it would be the case that $0 < u < \delta$, $0 < v < \delta$ and $f(u) - f(v) = 2$. But it would follow from the inequalities $0 < u < \delta$ and $0 < v < \delta$ that $f(u) - f(v) < \frac{1}{2}$, and thus a contradiction would arise were the limit of $f(x)$ as x tends to zero to exist. Therefore no such limit can exist.

Example Consider the function $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined such that

$$g(x) = 3x \sin\left(\frac{\pi}{2x}\right)$$

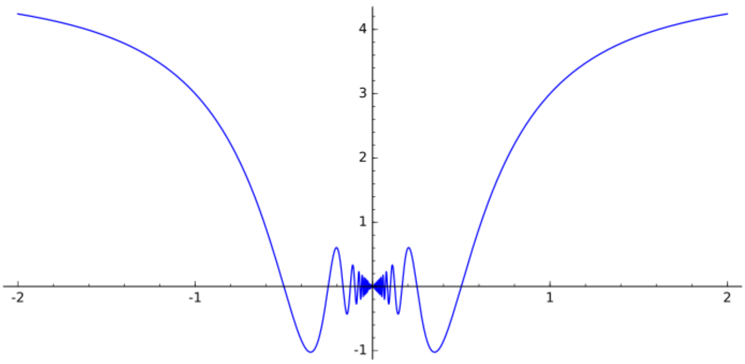
for all non-zero real numbers x. Now

$$-1 \leq \sin\left(\frac{\pi}{2x}\right) \leq 1$$

for all non-zero real numbers x. It follows that

$$-3|x| \leq g(x) \leq 3|x|$$

for all non-zero real numbers x.



Graph of $3x \sin\left(\dfrac{\pi}{2x}\right)$ as a function of x for $x \neq 0$.

(Plot generated using SageMath.)

Let some strictly positive real number ε be given. Let $\delta = \frac{1}{3}\varepsilon$. If a non-zero real number x satisfies

$$-\delta < x < \delta$$

then

$$-3\delta < g(x) < 3\delta,$$

and thus

$$-\varepsilon < g(x) < \varepsilon.$$

We conclude that

$$\lim_{x \to 0} g(x) = 0.$$

Example Consider the function $h\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined such that

$$h(x) = 2\sqrt{|x|}\,\sin\left(\frac{\pi}{2x}\right)$$

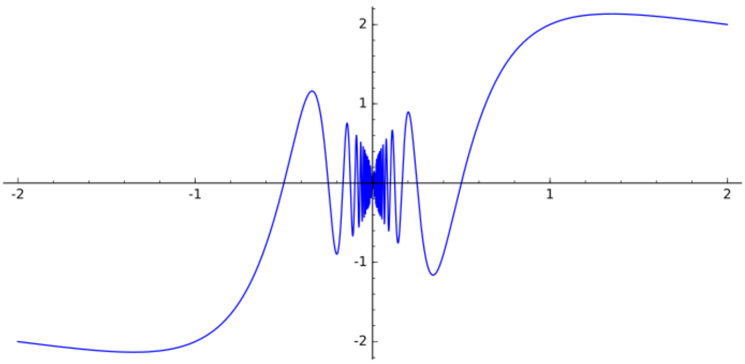
for all non-zero real numbers x. Now

$$-1 \leq \sin\left(\frac{\pi}{2x}\right) \leq 1$$

for all non-zero real numbers x. It follows that

$$-2\sqrt{|x|} \leq h(x) \leq 2\sqrt{|x|}$$

for all non-zero real numbers x.



Graph of $2\sqrt{|x|}\sin\left(\frac{\pi}{2x}\right)$ as a function of x for $x \neq 0$.

(Plot generated using SageMath.)

Let some strictly positive real number ε be given. Let $\delta = \frac{1}{4}\varepsilon^2$. If a non-zero real number x satisfies

$$-\delta < x < \delta$$

then

$$-2\sqrt{\delta} < h(x) < 2\sqrt{\delta},$$

and thus

$$-\varepsilon < h(x) < \varepsilon.$$

We conclude that

$$\lim_{x\to 0} h(x) = 0.$$

4.3 Limits of Polynomial Functions

Proposition 4.2 *Let $p(x)$ be a polynomial and let s be a real number. Then*

$$\lim_{x\to s} p(x) = p(s).$$

Proof It follows from the Remainder Theorem (Theorem 2.6) that

$$p(x) = (x - s)q_s(x) + p(s),$$

where $q_s(x)$ is the polynomial obtained by dividing the polynomial $p(x)$ by the polynomial $x - s$, taking quotient $q_s(x)$ and remainder $p(s)$.

Let n be the degree of the polynomial p, and let

$$q_s(x) = b_0 + b_1 x + b_2 x^2 + \cdots b_{n-1} x^{n-1}$$

where the coefficients $b_0, b_1, \ldots, b_{n-1}$ are real numbers. Let R be a real number chosen so that $R \geq 1$ and $-R < s < R$, and let C be a positive real number chosen large enough to ensure that

$$-C \leq b_j \leq C$$

for $j = 0, 1, \ldots, n-1$. Then

$$-CR^n \leq -CR^j \leq b_j x^j \leq CR^j \leq CR^n$$

for $j = 0, 1, \ldots, n-1$ and for all real numbers x satisfying $-R \leq x \leq R$. It follows that

$$-nCR^n \leq q_s(x) \leq nCR^n$$

for all real numbers x satisfying $-R \leq x \leq R$.

Let some strictly positive real number ε be given. Then some positive real number δ can be chosen so as to ensure that

$$-R \leq s - \delta < s + \delta \leq R$$

and $nCR^n\delta \leq \varepsilon$. If a real real number x satisfies $s - \delta < x < s + \delta$ then $-nCR^n \leq q_s(x) \leq nCR^n$ and therefore

$$-\varepsilon \leq -nCR^n\delta < (x - s)q_s(x) < nCR^n\delta \leq \varepsilon.$$

But $(x - s)q_s(x) = p(x) - p(s)$. We have thus shown that

$$p(s) - \varepsilon < p(x) < p(s) + \varepsilon$$

for all real numbers x satisfying $s - \delta < x < s + \delta$. It follows that

$$\lim_{x \to s} p(x) = p(s),$$

as required. ∎

4.4 Sums of Geometric Sequences

We prove some well-known formulae concerning finite sums of geometric sequences.

Proposition 4.3 *Let x be a real number, where $x \neq 1$. Then*

$$\sum_{j=0}^{n-1} x^j = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

for all natural numbers n.

Proof The identity

$$\sum_{j=0}^{n-1} x^j = \frac{x^n - 1}{x - 1}$$

is satisfied when $n = 1$. Indeed both sides of the identity have the value 1 when $n = 1$.

Let k be a natural number for which

$$\sum_{j=0}^{k-1} x^j = \frac{x^k - 1}{x - 1}.$$

Then

$$\begin{aligned}
\sum_{j=0}^{k} x^j &= \sum_{j=0}^{k-1} x^j + x^k = \frac{x^k - 1}{x - 1} + x^k \\
&= \frac{x^k - 1}{x - 1} + \frac{x^k(x-1)}{x - 1} = \frac{x^k - 1}{x - 1} + \frac{x^{k+1} - x^k}{x - 1} \\
&= \frac{x^{k+1} - 1}{x - 1}.
\end{aligned}$$

Thus if the identity

$$\sum_{j=0}^{n-1} x^j = \frac{x^n - 1}{x - 1}$$

holds when $n = k$, where k is some natural number, then this identity holds when $n = k + 1$. It follows from the Principle of Mathematical Induction that this identity follows for all natural numbers n, as required. ∎

Corollary 4.4 *Let n be a positive integer, and let u and v be distinct real numbers. Then*

$$\begin{aligned}
\frac{v^n - u^n}{v - u} &= u^{n-1} + u^{n-2}v + u^{n-2}v^2 + \cdots + uv^{n-2} + v^{n-1} \\
&= \sum_{j=0}^{n-1} u^{n-1-j} v^j.
\end{aligned}$$

Proof Let $x = \dfrac{v}{u}$. Then $x \neq 1$, because $u \neq v$, and $v^j = u^j x^j$ for all non-negative integers j. In particularly, $v - u = ux - u = u(x-1)$ and $v^n - u^n = u^n(x^n - 1)$. It follows from Proposition 4.3 that

$$\frac{v^n - u^n}{v - u} = \frac{u^{n-1}(x^n - 1)}{x - 1} = u^{n-1}\left(\sum_{j=0}^{n-1} x^j\right) = \sum_{j=0}^{n-1} u^{n-1-j} v^j,$$

as required. ∎

4.5 Derivatives of Polynomial Functions

Proposition 4.5 *Let n be a positive integer, and let s be a real number. Then*

$$\lim_{x \to s} \frac{x^n - s^n}{x - s} = ns^{n-1}.$$

Proof Let

$$q_s(x) = \sum_{j=0}^{n-1} s^{n-1-j} x^j.$$

for all real numbers x. Then $q_s(x)$ is a polynomial function of x. Moreover

$$\frac{x^n - s^n}{x - s} = q_s(x)$$

for all real numbers x distinct from s (see Corollary 4.4). Now

$$\lim_{x \to s} q_s(x) = q_s(s) = ns^{n-1},$$

because $q_s(x)$ is a polynomial function of x (see Proposition 4.2). It follows that

$$\lim_{x \to s} \frac{x^n - s^n}{x - s} = ns^{n-1},$$

as required. ∎

The result and proof strategy of Proposition 4.5 can be generalized to obtain the derivative

$$\lim_{x \to s} \frac{p(x) - p(s)}{x - s}$$

of a polynomial function $p(x)$ at a particular value s of the real variable x.

Proposition 4.6 *Let $p(x)$ be a polynomial function of x, and let*

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n.$$

Let s be a real number. Then

$$\lim_{x \to s} \frac{p(x) - p(s)}{x - s} = a_1 + 2a_2 s + 3a_3 s^2 + \cdots + n a_n s^{n-1}$$

Proof For each integer k between 1 and n, the kth power x^k of the real variable x satisfies the identity

$$\frac{x^k - s^k}{x - s} = q_{s,k}(x),$$

for all real numbers x distinct from x, where

$$q_{x,k}(x) = \sum_{j=0}^{k-1} s^{k-1-j} x^j$$

(see Corollary 4.4). Multiplying the identity satisfied by x^k by a_k and summing for $k = 1, 2, \ldots, n$, we find that

$$\frac{p(x) - p(s)}{x - s} = q_s(x),$$

where

$$q_s(x) = \sum_{k=1}^{n} a_k q_{s,k}(x).$$

Now $q_{s,k}(x)$ is a polynomial function of x for $k = 1, 2, \ldots, n$. It follows that $q_s(x)$ is a polynomial function of x, and therefore

$$\begin{aligned} \lim_{x \to s} q_s(x) &= q_s(s) = \sum_{k=1}^{n} a_k q_{s,k}(s) \\ &= \sum_{k=1}^{n} k a_k s^{k-1}. \end{aligned}$$

The result follows. ∎

Let $p(x)$ be a polynomial function of a real variable x. The *derivative* $p'(x)$ of the polynomial $p(x)$ is defined so that its value at a real number s satisfies

$$p'(s) = \lim_{x \to s} \frac{p(x) - p(s)}{x - s} = \lim_{h \to 0} \frac{p(s + h) - p(s)}{h}.$$

for all real numbers s. The derivative of $p(x)$ may also be denoted by the expressions

$$\frac{dp(x)}{dx} \qquad \frac{d}{dx}(p(x)).$$

Proposition 4.6 shows that if

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

then

$$p'(x) = a_1 + 2a_2 s + 3a_3 s^2 + \cdots + n a_n s^{n-1}$$

Thus, for example, if

$$p(x) = ax^3 + bx^2 + cx + d,$$

where a. b, c and d are real constants, then

$$p'(x) = 3ax^2 + 2bx + c.$$

4.6 Local Maxima and Minima of Polynomial Functions

Lemma 4.7 *Let $p(x)$ be a polynomial, let s be a real number, and let $p'(s)$ be the derivative of p at s. Suppose that $p'(s) > 0$. Then there exists a positive real number δ such that $p(x) > p(s)$ for all real numbers x satisfying $s < x < s + \delta$ and $p(x) < p(s)$ for all real numbers x satisfying $s - \delta < x < s$.*

Proof It follows from the Remainder Theorem (Theorem 2.6) that a polynomial $q(x)$ can be determined so that

$$p(x) = (x - s)q(x) + p(s).$$

Now $q(s) = p'(s)$ (see Proposition 4.6). Moreover $q(s) = \lim_{x \to s} q(x)$ (see Proposition 4.2). Now $q(s) = p'(s) > 0$. It follows the definition of limits that there exists some strictly positive real number δ so that $q(x) > 0$ for all positive real numbers x satisfying $s - \delta < x < s + \delta$. But then the equation

$$p(x) = (x - s)q(x) + p(s).$$

ensures that $p(x) > p(s)$ for all real numbers x satisfying $s < x < s + \delta$, and $p(x) < p(s)$ for all real numbers x satisfying $s - \delta < x < s$. The result follows. ∎

Proposition 4.8 *Let $p(x)$ be a polynomial, let s be a real number, and let $p'(s)$ be the derivative of p at s. Suppose that the function $x \mapsto p(x)$ mapping each real number x to $p(x)$ has a local maximum or local minimum at $x = s$. Then $p'(s) = 0$.*

Proof It follows from Lemma 4.7 that if $p'(s) > 0$ then the function $x \mapsto p(x)$ cannot have a local maximum or local minimum at $x = s$. Applying this result with p replaced by $-p$, we see that if $p'(s) < 0$ then $(-p)'(s) > 0$, and therefore the function $x \mapsto -p(x)$ cannot have a local maximum or local minimum at $x = s$. It follows that if $p'(s) < 0$ then the function $x \mapsto p(x)$ itself cannot have a local maximum or local minimum at $x = s$. Thus if the function $x \mapsto p(x)$ *does* have a local maximum or local minimum at $x = s$, then the only remaining possibility is that $p'(s) = 0$. The result follows. ∎

Example Let

$$p(x) = x^3 - 9x^2 + 24x - 16.$$

Then

$$\begin{aligned} p'(x) &= 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) \\ &= 3(x-2)(x-4). \end{aligned}$$

It follows that the local maxima and minima must be located at $x = 2$ and $x = 4$, and indeed $p(x)$ achieves a local minimum with value 0 at $x = 4$ and a local maximum with value 4 at $x = 2$.

4.7 Absolute Values of Real Numbers

Let x be a real number. The *absolute value* $|x|$ of x is defined so that

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0; \end{cases}$$

Lemma 4.9 *Let x and y be real numbers. Then $|x + y| \leq |x| + |y|$ and $|xy| = |x|\,|y|$.*

Proof Let x and y be real numbers. Then

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|.$$

On adding inequalities, we find that

$$-(|x| + |y|) = -|x| - |y| \leq x + y \leq |x| + |y|,$$

and thus

$$x + y \le |x| + |y| \quad \text{and} \quad -(x + y) \le |x| + |y|.$$

Now the value of $|x + y|$ is equal to at least one of the numbers $x + y$ and $-(x + y)$. It follows that

$$|x + y| \le |x| + |y|$$

for all real numbers x and y.

Next we note that $|x|\,|y|$ is the product of one or other of the numbers x and $-x$ with one or other of the numbers y and $-y$, and therefore its value is equal either to xy or to $-xy$. Because both $|x|\,|y|$ and $|xy|$ are non-negative, we conclude that $|xy| = |x|\,|y|$, as required. ∎

Absolute values can be used in order to express some of the inequalities occurring in formal definitions of mathematical concepts such as limits, continuity and convergence in more compact form.

Consider the formal definition of limits. For simplicity we consider a real-valued function $f\colon \mathbb{R} \setminus \{s\} \to \mathbb{R}$ defined over the set $\mathbb{R} \setminus \{s\}$ of real numbers distinct from s. Let L be a real number. Then

$$\lim_{x \to s} f(x) = L$$

if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

for all real numbers x that satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \ne s.$$

So let us first consider inequalities of the form

$$L - \varepsilon < y < L + \varepsilon,$$

where y, L and ε are real numbers and $\varepsilon > 0$. Subtracting L from both sides and using the definition of absolute values, we find that

$$\begin{aligned} & L - \varepsilon < y < L + \varepsilon \\ \iff\quad & -\varepsilon < y - L < \varepsilon \\ \iff\quad & |y - L| < \varepsilon \end{aligned}$$

(Here $\iff$ signifies "if and only if".)

Thus the condition on the values of the function f requiring that

$$L - \varepsilon < f(x) < L + \varepsilon$$

for values of real variable x satisfying appropriate constraints is completely equivalent to the condition that

$$|f(x) - L| < \varepsilon$$

for those values of x. Both forms of this condition express the fact that the quantity $f(x)$ is within a distance ε of the limiting value L.

This analysis shows that the function $f: \mathbb{R}\setminus\{s\} \to \mathbb{R}$ satisfies $\lim_{x\to s} f(x) = L$ if and only if, given any positive real number ε, there exists a positive real number δ such that

$$|f(x) - L| < \varepsilon$$

for all real numbers x that satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

(Note that, according to standard definitions, positive real numbers are required to be non-zero. Thus the terms "positive" and "strictly positive" are synonymous.)

Next we examine the constraints on the real variable x. This is required to satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

Now, from what has already been shown, we see that

$$s - \delta < x < s + \delta$$

if and only if $|x - s| < \delta$. But we also need that $x \neq s$. This condition is equivalent to requiring that $|x - s| > 0$. It follows that both conditions

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s$$

are satisfied simultaneously if and only if

$$0 < |x - s| < \delta.$$

We conclude that the function $f: \mathbb{R} \setminus \{s\} \to \mathbb{R}$ satisfies $\lim_{x\to s} f(x) = L$ if and only if, given any positive real number ε, there exists a positive real number δ such that

$$|f(x) - L| < \varepsilon$$

for all real numbers x that satisfy

$$0 < |x - s| < \delta.$$

Example Let s be a positive real number. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined such that

$$f(x) = \sqrt{1 + 3x^2}.$$

for all real numbers x. Now $f(1) = 2$. We investigate whether this function tends to the limit 2 as x tends to 1.

Now $\lim_{x \to 1} f(x) = 2$ if and only if, given any positive real number ε, there exists some positive real number δ such that $|f(x) - 2| < \varepsilon$ for all real numbers x that satisfy $0 < |x - 1| < \delta$. Now $f(x)$ is an increasing function of x when $x > 0$. Taking account of this, we see that, given $\varepsilon > 0$, it suffices to discover a positive real number δ so that

$$f(1 + \delta) \leq 2 + \varepsilon$$

and

$$f(1 - \delta) \geq 2 - \varepsilon.$$

Indeed, because $f(x)$ increases with x when $x > 0$, if a value of δ is found that satisfies these inequalities then

$$2 - \varepsilon \leq f(1 - \delta) < f(x) < f(1 + \delta) \leq 2 + \varepsilon$$

Now

$$\begin{aligned} & f(1 + \delta) \leq 2 + \varepsilon \\ \iff & \sqrt{1 + 3(1 + \delta)^2} \leq 2 + \varepsilon \\ \iff & 1 + 3(1 + \delta)^2 \leq (2 + \varepsilon)^2 \\ \iff & 4 + 6\delta + 3\delta^2 \leq 4 + 4\varepsilon + \varepsilon^2 \\ \iff & 6\delta + 3\delta^2 \leq 4\varepsilon + \varepsilon^2. \end{aligned}$$

There is another inequality that needs to be satisfied, namely the inequality

$$f(1 - \delta) \geq 2 - \varepsilon.$$

We investigate what this inequality entails.

$$\begin{aligned} & f(1 - \delta) \geq 2 - \varepsilon \\ \iff & \sqrt{1 + 3(1 - \delta)^2} \geq 2 - \varepsilon \\ \iff & 1 + 3(1 - \delta)^2 \geq (2 - \varepsilon)^2 \\ \iff & 4 - 6\delta + 3\delta^2 \geq 4 - 4\varepsilon + \varepsilon^2 \\ \iff & -6\delta + 3\delta^2 \geq -4\varepsilon + \varepsilon^2 \\ \iff & 6\delta - 3\delta^2 \leq 4\varepsilon - \varepsilon^2. \end{aligned}$$

It follows that the positive real number δ must be chosen so as to ensure that

$$6\delta + 3\delta^2 \leq 4\varepsilon + \varepsilon^2 \quad \text{and} \quad 6\delta - 3\delta^2 \leq 4\varepsilon - \varepsilon^2.$$

Having arrived at this stage, there is certainly more than one way to find a positive real number δ that satisfies these inequalities. We shall proceed by solving the quadratic equations that determine positive real numbers δ_1 and δ_2 for which

$$3\delta_1^2 + 6\delta_1 - 4\varepsilon - \varepsilon^2 = 0 \quad \text{and} \quad -3\delta_2^2 + 6\delta_2 - 4\varepsilon + \varepsilon^2 = 0.$$

The quadratic formula yields the results that

$$\begin{aligned} \delta_1 &= \frac{1}{3}(-3 + \sqrt{9 + 12\varepsilon + 3\varepsilon^2}) \\ \delta_2 &= \frac{1}{3}(3 - \sqrt{9 - 12\varepsilon + 3\varepsilon^2}) \end{aligned}$$

(The solutions resulting from application of the quadratic formula have been discarded, as δ must be close to zero when ε is close to zero. The discarded solutions correspond to values of δ_1 and δ_2 that bring $1 + \delta_1$ and $1 - \delta_2$ close to -1 for small ε). If we now take δ to be the minimum of δ_1 and δ_2 then $\delta > 0$ and $|f(x) - 2| < \varepsilon$ for all real numbers x that satisfy $0 < |x - 1| < \delta$.

4.8 Limit Points of Sets of Real Numbers

There is a technicality that needs to be addressed in order to formulate a definition of limit of a real-valued function defined over a subset D of the set $\mathbb{R}$ of real numbers that has the generality required in order to make use of the theory of limits to define concepts such as continuity and one-sided limits.

Suppose that we have a real-valued function $f\colon D \to \mathbb{R}$ defined over some subset D of the set of real numbers. For which real numbers s is it appropriate to consider whether or not the limit $\lim_{x \to s} f(x)$ exists?

Example Let $f\colon [-1, 1] \to \mathbb{R}$ be the function defined over the interval $[-1, 1]$ so that $f(x) = \sqrt{1 - x^2}$, where

$$[-1, 1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}.$$

It would make no sense to consider what value, if any, could be the limit $\lim_{x \to 100}$ of $f(x)$ as x tends to 100.

Example Let $f\colon (-1,1) \to \mathbb{R}$ be the function defined over the interval $(-1,1)$ so that

$$f(x) = \frac{(x - x^2)\sqrt{1 - x^2}}{1 - x}.$$

It might well make sense to consider what value, if any, could be the limit $\lim_{x\ to 1}$ of $f(x)$ as x tends to 1, even though the number 1 lies outside the domain of the function. And indeed the function $f(x)$ tends to zero as x tends to 1.

The appropriate concept that determines those values of s at which limits can sensibly be taken is that of a *limit point*. This concept enters into the statement of formal propositions concerning limits, but its primary purpose is to ensure that limits, determined according to the formal definition of limits, are in fact uniquely determined according to that definition. Thus although it appears regularly in the statement of propositions, the defining property of limit points rarely appears in proofs.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers. A real number s is a *limit point* of D if, given any positive real number δ, there exists a real number x belonging to D which satisfies $0 < |x - s| < \delta$.

To summarize: if $f\colon D \to \mathbb{R}$ is a real-valued function defined over a subset D of $\mathbb{R}$, and if s is a limit point of D, then it makes sense to consider whether or not the limit $\lim_{x \to s} f(x)$ has a well-defined value as x tends to s in D. If s is not a limit point of D then it makes no sense to consider whether or not this limit has a well-defined value.

4.9 Limits of Functions of a Real Variable

The following definition is the standard definition of limits of real-valued functions defined over subsets of the set $\mathbb{R}$ of real numbers.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point belonging to D, and let L be a real number. The real number L is said to be the *limit* of $f(x)$, as x tends to s in D, if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - L| < \varepsilon$ for all real numbers x in D that satisfy $0 < |x - s| < \delta$.

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point belonging to D, and let L be a real number. If L is the limit of $f(x)$ as x tends to s in D then we can denote this fact by writing $\lim_{x \to s} f(x) = L$.

Note that the inequality $|f(x) - L| < \varepsilon$ is satisfied at x, where $x \in D$, if and only if

$$L - \varepsilon < f(x) < L + \varepsilon.$$

Also the inequality $0 < |x - s| < \delta$ is satisfied if and only if both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$

Lemma 4.10 *Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset $\mathbb{D}$ of $\mathbb{R}$, let s be a limit point of D, and let L be a real number. Suppose that $\lim_{x \to s} f(x) = L$. Then $\lim_{x \to s}(-f(x)) = -L$.*

Proof Let some positive real number ε be given. Then there exists some positive real number δ such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

whenever $0 < |x - s| < \delta$. Taking the negatives of the quantities satisfying the above inequalities we see that

$$-L - \varepsilon < -f(x) < -L + \varepsilon$$

whenever $0 < |x - s| < \delta$. We conclude that $\lim_{x \to s}(-f(x)) = -L$, as required. ∎

Lemma 4.11 *Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset $\mathbb{D}$ of $\mathbb{R}$, let s be a limit point of D, and let L and c be real numbers. Suppose that $\lim_{x \to s} f(x) = L$. Then $\lim_{x \to s}(f(x) + c) = L + c$.*

Proof Let some positive real number ε be given. Then there exists some positive real number δ such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

whenever $0 < |x - s| < \delta$. Adding the constant c to all terms in these inequalities, we see that

$$L + c - \varepsilon < f(x) + c < L + c + \varepsilon$$

whenever $0 < |x - s| < \delta$. We conclude that $\lim_{x \to s}(f(x) + c) = L + c$, as required. ∎

Lemma 4.12 *Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset $\mathbb{D}$ of $\mathbb{R}$, let s be a limit point of D, and let L and M be real numbers where $M \neq 0$. Suppose that $\lim_{x \to s} f(x) = L$. Then $\lim_{x \to s}(Mf(x)) = ML$.*

Proof In view of Lemma 4.10 we need only consider the case when $M > 0$. Let some positive real number ε be given. Then some positive real number ε_0 can be chosen small enough to ensure that $M\varepsilon_0 < \varepsilon$. Then there exists some positive real number δ such that

$$L - \varepsilon_0 < f(x) < L + \varepsilon_0$$

whenever $0 < |x - s| < \delta$. But then

$$ML - \varepsilon < ML - M\varepsilon_0 < Mf(x) < ML + M\varepsilon_0 < ML + \varepsilon$$

whenever $0 < |x - s| < \delta$. We conclude that $\lim_{x \to s}(Mf(x)) = ML$, as required. ∎

Proposition 4.13 *Let D be a subset of $\mathbb{R}$, let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be real-valued functions on D, let s be a limit point of D, and let L and M be real numbers. Suppose that*

$$\lim_{x \to s} f(x) = L$$

and

$$\lim_{x \to s} g(x) = M.$$

Then

$$\lim_{x \to s}(f(x) + g(x)) = L + M.$$

Proof Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$L - \tfrac{1}{2}\varepsilon < f(x) < L + \tfrac{1}{2}\varepsilon$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta_1$ and

$$M - \tfrac{1}{2}\varepsilon < g(x) < M + \tfrac{1}{2}\varepsilon,$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if a real number x in D satisfies $0 < |x - s| < \delta$ then

$$L - \tfrac{1}{2}\varepsilon < f(x) < L + \tfrac{1}{2}\varepsilon$$

and

$$M - \tfrac{1}{2}\varepsilon < g(x) < M + \tfrac{1}{2}\varepsilon,$$

and therefore

$$L + M - \varepsilon < f(x) + g(x) < L + M + \varepsilon.$$

It follows that

$$\lim_{x \to s} (f(x) + g(x)) = L + M,$$

as required. ∎

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a limit point of D. Let $f\colon D \to \mathbb{R}$ be a real-valued function on D. We say that $f(x)$ *remains bounded* as x tends to s in D if there exist strictly positive constants C and δ such that $-C \leq f(x) \leq C$ for all real numbers x in D that satisfy $0 < |x - s| < \delta$.

Proposition 4.14 *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued function on some subset D of $\mathbb{R}$, and let s be a limit point of D. Suppose that $\lim_{x \to s} f(x) = 0$. Suppose also that $g(x)$ remains bounded as x tends to s in D. Then*

$$\lim_{x \to s} \Bigl(f(x) g(x) \Bigr) = 0.$$

Proof Let some strictly positive real number ε be given. Then $g(x)$ remains bounded as x tends to s in D, and therefore positive constants C and δ_0 can be determined so that $-C \leq g(x) \leq C$ for all $x \in D$ satisfying $0 < |x - s| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(x)| < \varepsilon_0$ whenever $0 < |x - s| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |x - s| < \delta$ then $|g(x)| \leq C$ and $|f(x)| < \varepsilon_0$, and therefore

$$|f(x) g(x)| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.15 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ be a function mapping D into $\mathbb{R}$, let $g\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of D, let L and M be real numbers. Suppose that*

$$\lim_{x \to s} f(x) = L$$

and

$$\lim_{x \to s} g(x) = M.$$

Then

$$\lim_{x \to s} f(x) g(x) = LM.$$

Proof The functions f and g satisfy the equation

$$f(x)g(x) = g(x)\Big(f(x) - L\Big) + (g(x) - M)L + LM,$$

where

$$\lim_{x \to s}\Big(f(x) - L\Big) = 0 \quad \text{and} \quad \lim_{x \to s}\Big(g(x) - M\Big) = 0.$$

Moreover there exists a strictly positive constant δ_0 such that

$$M - 1 < g(x) < M + 1$$

for all $x \in D$ satisfying $0 < |x - s| < \delta_0$. Thus the function g remains bounded as x tends to s in D. It now follows that

$$\lim_{x \to s} (g(x)(f(x) - L)) = 0$$

(see Proposition 4.14). Similarly

$$\lim_{x \to s} (g(x) - M))\, L = 0.$$

It follows that

$$\begin{aligned} & \lim_{x \to s} (f(x)g(x)) \\ = \; & \lim_{x \to s} (g(x)(f(x) - L)) + \lim_{x \to s} \Big(\Big(g(x) - M\Big)L\Big) + LM \\ = \; & LM, \end{aligned}$$

as required. ∎

Lemma 4.16 *Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set $\mathbb{R}$ of real numbers, let s be a limit point of D, and let L be a real number. Suppose that $\lim_{x \to s} f(x) = L$, where $L \neq 0$.*

$$\lim_{x \to s} \frac{1}{f(x)} = \frac{1}{L}.$$

Proof We first prove the result in the case when $L > 0$. In this case we can choose a constant c such that $0 < c < L$. (For example, we could choose $c = \frac{1}{2}L$.) Now

$$\frac{1}{f(x)} - \frac{1}{L} = \frac{L - f(x)}{Lf(x)}.$$

It follows that

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| < \frac{1}{c^2}|f(x) - L|$$

whenever $f(x) \geq c$.

Now let some positive real number ε be given. Then a positive real number ε_0 can be found which is small enough to ensure that both $L - \varepsilon_0 \geq c$ and $0 < \varepsilon_0 \leq c^2\varepsilon$. (For example, we could take ε_0 to be the minimum of $L - c$ and ε/c^2.) Now $\lim_{x\to s} f(x) = L$. It therefore follows that there exists some positive real number δ that is small enough to ensure that

$$|f(x) - L| < \varepsilon_0$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta$. It follows that if x is a real number in D that satisfies $0 < |x - s| < \delta$ then

$$c \leq L - \varepsilon_0 < f(x)$$

and

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| < \frac{1}{c^2}|f(x) - L| < \frac{\varepsilon_0}{c^2} \leq \varepsilon.$$

We conclude from this that if $f\colon D \to \mathbb{R}$ satisfies $\lim_{x\to s} f(x) = L$, where $L > 0$, then

$$\lim_{x\to s} \frac{1}{f(x)} = \frac{1}{L}.$$

Now suppose that satisfies $\lim_{x\to s} f(x) = L$, where $L < 0$, Then satisfies $\lim_{x\to s}(-f(x)) = -L$ where $-L > 0$ (see Lemma 4.10) and therefore

$$\lim_{x\to s} -\frac{1}{f(x)} = -\frac{1}{L}.$$

It follows that

$$\lim_{x\to s} \frac{1}{f(x)} = \frac{1}{L}$$

in this case also. This completes the proof. ∎

Proposition 4.17 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions on D, and let s be a limit point of the set D. Suppose that $\lim_{x\to s} f(x)$ and $\lim_{x\to s} g(x)$ both exist. Then so do $\lim_{x\to s}(f(x)+g(x))$, $\lim_{x\to s}(f(x)-g(x))$ and $\lim_{x\to s}(f(x)g(x))$, and moreover*

$$\begin{aligned}
\lim_{x\to s}(f(x) + g(x)) &= \lim_{x\to s} f(x) + \lim_{x\to s} g(x),\\
\lim_{x\to s}(f(x) - g(x)) &= \lim_{x\to s} f(x) - \lim_{x\to s} g(x),\\
\lim_{x\to s}(f(x)g(x)) &= \lim_{x\to s} f(x) \times \lim_{x\to s} g(x),
\end{aligned}$$

If moreover $g(x) \neq 0$ *for all* $x \in D$ *and* $\lim_{x \to s} g(x) \neq 0$ *then*

$$\lim_{x \to s} \frac{f(x)}{g(x)} = \frac{\lim_{x \to s} f(x)}{\lim_{x \to s} g(x)}.$$

Proof It follows from Proposition 4.13 (applied in the case when the target space is one-dimensional) that

$$\lim_{x \to s} (f(x) + g(x)) = \lim_{x \to s} f(x) + \lim_{x \to s} g(x).$$

Replacing the function g by $-g$, we deduce that

$$\lim_{x \to s} (f(x) - g(x)) = \lim_{x \to s} f(x) - \lim_{x \to s} g(x).$$

It follows from Proposition 4.15 that

$$\lim_{x \to s} (f(x)g(x)) = \lim_{x \to s} f(x) \times \lim_{x \to s} g(x).$$

Now suppose that $g(x) \neq 0$ for all $x \in D$ and that $\lim_{x \to s} g(x) \neq 0$. It follows from Lemma 4.16 that

$$\lim_{x \to s} \frac{1}{g(x)} = \frac{1}{\lim_{x \to s} g(x)}.$$

It then follows from Proposition 4.15 that

$$\lim_{x \to s} \frac{f(x)}{g(x)} = \frac{\lim_{x \to s} f(x)}{\lim_{x \to s} g(x)}.$$

This completes the proof. ∎

Proposition 4.18 *Let* $f\colon D \to \mathbb{R}$ *be a real-valued function defined over a subset* D *of the set* $\mathbb{R}$ *of real numbers, let* s *be a limit point of* D*, and let* L *be a real number. Suppose that* $f(x) \geq 0$ *for all real numbers* x *in* D *that satisfy* $x \neq s$*. Suppose also that* $\lim_{x \to s} f(x) = L$*. Then* $L \geq 0$*.*

Proof Suppose that it were the case that $L < 0$. Let $\varepsilon = -L$. Then $\varepsilon > 0$. Therefore there would exist some positive real number δ such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

for all real numbers x belonging to D that satisfy $0 < |x - s| < \delta$. But $\varepsilon = -L$. It would therefore follow that $f(x) < 0$ for all real numbers x belonging to D that satisfy $0 < |x - s| < \delta$. Moreover there exists at least elements of D that satisfy these inequalities because s is a limit point of D. Thus the hypothesis that $L < 0$ results in a contradiction. It follows that $\lim_{x \to s} f(x) \geq 0$, as required. ∎

Corollary 4.19 *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions defined over a subset D of the set $\mathbb{R}$ of real numbers, and let s be a limit point of D. Suppose that $f(x) \leq g(x)$ for all $x \in D$, and that $\lim\limits_{x\to s} f(x)$ and $\lim\limits_{x\to s} g(x)$ both exist. Then*

$$\lim_{x\to s} f(x) \leq \lim_{x\to s} g(x).$$

Proof The inequality $g(x) - f(x) \geq 0$ is satisfied for all $x \in D$. It follows from Proposition 4.17 and Proposition 4.18 that

$$\lim_{x\to s} g(x) - \lim_{x\to s} f(x) = \lim_{x\to s}(g(x) - f(x)) \geq 0,$$

and thus $\lim\limits_{x\to s} f(x) \leq \lim\limits_{x\to s} g(x)$, as required. ∎

Theorem 4.20 (Squeeze Theorem) *Let f, g and h be real-valued functions defined over a subset D of the set $\mathbb{R}$ of real numbers, let s be a limit point of D, and let L be a real number. Suppose that $f(x) \leq g(x) \leq h(x)$ for all real numbers x satisfying $x \neq s$ that belong to D. Suppose also that*

$$\lim_{x\to s} f(x) = \lim_{x\to s} h(x) = L,$$

so that the real number L is the limit both of $f(x)$ and of $h(x)$ as x tends to s in D. Then

$$\lim_{x\to s} g(x) = L.$$

Proof Let some positive real number ε be given. Then there exist positive real numbers δ_1 and δ_2 such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

whenever $0 < |x - s| < \delta_1$ and

$$L - \varepsilon < h(x) < L + \varepsilon$$

whenever $0 < |x - s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if a real number x belonging to D satisfies $0 < |x - s| < \delta$ then

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon,$$

and therefore $\lim\limits_{x\to s} g(x) = L$, as required. ∎

Example Let $g: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined so that

$$g(x) = 3\sqrt{|x|}\sin\left(\frac{\pi}{2x}\right)$$

for all non-zero real numbers x. Then

$$f(x) \leq g(x) \leq h(x)$$

for all non-zero real numbers x, where

$$f(x) = -3\sqrt{|x|} \quad \text{and} \quad h(x) = 3\sqrt{|x|}$$

for all real numbers x.

Now, given any positive real number ε, a positive real number δ could be chosen such that $3\sqrt{\delta} \leq \varepsilon$. For example, one could choose $\delta = \frac{1}{9}\varepsilon^2$. Then $-\varepsilon < f(x) \leq h(x) < \varepsilon$ for all real numbers x satisfying $0 < |x| < \delta$. We have thus shown that

$$\lim_{x \to 0} f(x) = 0 \quad \text{and} \quad \lim_{x \to 0} h(x) = 0.$$

It follows from the Squeeze Theorem (Theorem 4.20) that $\lim_{x \to 0} g(x) = 0$.

4.10 Continuity

The concept of *continuity* for functions of a real variable is defined formally as follows.

Definition Let $f: D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set of real numbers, and let s be a real number belonging to D. The function f is *continuous* at s if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$.

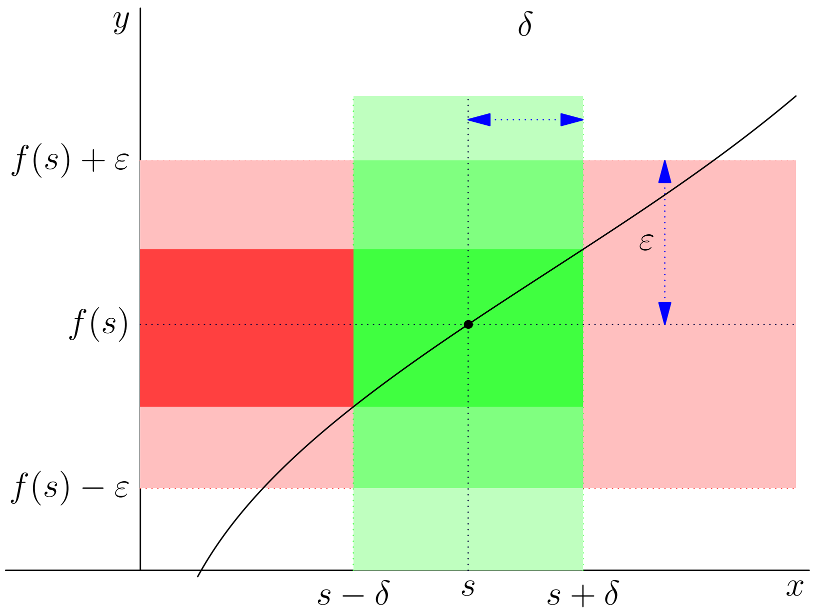
A real-valued function $f: D \to \mathbb{R}$ is said to be continuous on D if it is continuous at every real number belonging to D.

The definition of continuity can be expressed as follows: a real-valued function $f: D \to \mathbb{R}$ defined on a subset D of the set of real numbers is continuous at s, where $s \in D$, if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that

$$f(s) - \varepsilon < f(x) < f(s) + \varepsilon$$

for all real numbers x belonging to D that satisfy

$$s - \delta < x < s + \delta.$$



Example The function $f\colon \mathbb{R} \to \mathbb{R}$ defined such that $f(x) = x^3$ for all real numbers x is continuous. Indeed let s be a real number. Then

$$f(x) - f(s) = x^3 - s^3 = (x - s)(x^2 + xs + s^2).$$

Let $B = |s| + 1$. If x is a real number satisfying $s - 1 < x < s + 1$ then $-B \leq x \leq B$ and therefore

$$-3B^2 \leq x^2 + xs + s^2 \leq 3B^2.$$

It follows that

$$|f(x) - f(s)| \leq 3B^2 |x - s|$$

for all real numbers x satisfying $s - 1 < x < s + 1$.

Now let some positive real number ε be given. Then some positive real number δ can be chosen small enough to ensure that both $0 < \delta < 1$ and $\delta \leq \dfrac{\varepsilon}{3B^2}$. It then follows that if x is any real number satisfying $|x - s| < \delta$ then $-B \leq s - 1 < x < s + 1 \leq B$, and therefore

$$|f(x) - f(s)| \leq 3B^2 |x - s| < 3B^2 \delta \leq \varepsilon.$$

We have therefore verified that the formal definition of continuity is satisfied by the function f.

The definition of continuity is obviously closely related to the definition of limits. Indeed, examining definitions, we see that if $f\colon D \to \mathbb{R}$ is a real-valued function defined on a subset D of the set of real numbers, and if s is a real number belonging to D that is also a limit point of D, then the function f is continuous at s if and only if $\lim_{x \to s} f(x) = f(s)$.

The following proposition states a necessary and sufficient condition for a function $f\colon D \to \mathbb{R}$ to be continuous on D.

Proposition 4.21 *Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers. Then f is continuous on D if and only if $\lim_{x \to s} f(x) = f(s)$ for all real numbers s belonging to D that are limit points of D.*

Proof It follows from the definitions of limits and continuity that the function f is continuous at a real number s belonging to D that is also a limit point of D if and only if $\lim_{x \to s} f(x) = f(s)$.

If s is a real number belonging to D that is not a limit point of D then it follows from the definition of limit points that there exists some strictly positive real number δ for which

$$\{x \in D \mid |x - s| < \delta\} = \{s\}.$$

It then follows that $|f(x) - f(s)| = 0$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$, because the only real number x satisfying this inequality is s itself. It follows that the function f is continuous at any point of D that is not a limit point of D. The result follows. ∎

The following result follows immediately from Proposition 4.21 Proposition 4.17.

Proposition 4.22 *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be continuous functions defined over a subset D of the set of real numbers. Suppose that the functions f and g are continuous on D. Then the function $f + g$, $f - g$ and $f \cdot g$ are continuous on D, where $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$ and $(f \cdot g)(x) = f(x)g(x)$ for all real numbers x belonging to D. Moreover if the function g is non-zero throughout D then the function f/g is continuous on D, where $(f/g)(x) = f(x)/g(x)$ for all real numbers x belonging to D.*

The following result follows immediately from Proposition 4.2 and Proposition 4.21. It can also be deduced through a straightforward application of Proposition 4.22

Proposition 4.23 *All polynomial functions are continuous.*

Example We determine whether or not

$$\lim_{x\to 0} \frac{6x^2 + 8x^3 + 7x^5}{3x^2 + 8x^4 + x^7}$$

exists, and, if so, what is the value of the limit. Now the value of the limit of this expression at $x = 0$ is determined by the values of the expression for non-zero values of x. And

$$\frac{6x^2 + 8x^3 + 7x^5}{3x^2 + 8x^4 + x^7} = \frac{6 + 8x + 7x^3}{3 + 8x^2 + x^5}$$

when $x \neq 0$. Now the numerator and denominator of the fraction on the right hand side of the above equation are both polynomial functions. It follows that limit of these polynomial functions as x tends to zero is the value of the polynomials at $x = 0$ (see Proposition 4.2) Therefore

$$\lim_{x\to 0}(6 + 8x + 7x^3) = 6 \quad \text{and} \quad \lim_{x\to 0}(3 + 8x^2 + x^5) = 3.$$

It then follows from Proposition 4.22 that the limit of the given expression exists, and

$$\begin{aligned} \lim_{x\to 0} \frac{6x^2 + 8x^3 + 7x^5}{3x^2 + 8x^4 + x^7} &= \lim_{x\to 0} \frac{6 + 8x + 7x^3}{3 + 8x^2 + x^5} \\ &= \frac{\lim_{x\to 0}(6 + 8x + 7x^3)}{\lim_{x\to 0}(3 + 8x^2 + x^5)} \\ &= \frac{6}{3} = 2. \end{aligned}$$

Example We now determine the value, if it exists, of

$$\lim_{x\to 3} \frac{x^2 - 5x + 6}{x^2 + 2x - 15}.$$

In this case the numerator and denominator of the fraction are zero when $x = 3$. But both are quadratic polynomials which can be factored. Indeed

$$\frac{x^2 - 5x + 6}{x^2 + 2x - 15} = \frac{(x-3)(x-2)}{(x-3)(x+5)} = \frac{x-2}{x+5}$$

when $x \neq 3$. Moreover the numerator and denominator of the expression on the extreme right above are continuous functions of the variable x that are both non-zero when $x = 3$. It follows that

$$\lim_{x\to 3} \frac{x^2 - 5x + 6}{x^2 + 2x - 15} = \lim_{x\to 3} \frac{(x-3)(x-2)}{(x-3)(x+5)} = \lim_{x\to 3} \frac{x-2}{x+5} = \frac{3-2}{3+5} = \frac{1}{8}.$$

Proposition 4.24 *Let m and n be positive integers, and let s be a positive real number. Then*

$$\lim_{x\to s} x^{\frac{m}{n}} = s^{\frac{m}{n}}$$

and

$$\lim_{x\to s} \frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x - s} = \frac{m}{n} s^{\frac{m}{n}-1}.$$

Proof Let s and x be positive real numbers, and let $u = s^{\frac{1}{n}}$ and $v = x^{\frac{1}{n}}$. Then

$$\begin{aligned} \frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} &= \frac{v^m - u^m}{v - u} = \frac{u^m - v^m}{u - v} = \sum_{j=0}^{m-1} u^{m-1-j} v^j \\ &= \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} x^{\frac{j}{n}} \end{aligned}$$

(see Corollary 4.4).

The real number s is positive. We can therefore choose a positive real number A small enough to ensure that $0 < A^n < s$. If x satisfies $x > A^n$, and if $u = s^{\frac{1}{n}}$ and $v = x^{\frac{1}{n}}$ then $u > A$ and $v > A$. It follows that

$$\sum_{j=0}^{m-1} u^{m-1-j} v^j \geq \sum_{j=0}^{m-1} A^{m-1-j} A^j \geq \sum_{j=0}^{m-1} A^{m-1} = mA^{m-1}.$$

Thus

$$\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} \geq mA^{m-1}$$

whenever $s > A^n$ and $x > A^n$.

Applying this result in the case when $m = n$, we see that

$$\frac{x - s}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} \geq nA^{n-1},$$

and therefore

$$\frac{x^{\frac{1}{n}} - s^{\frac{1}{n}}}{x - s} \leq \frac{1}{nA^{n-1}}$$

for all real numbers x and s satisfying $s > A^n$ and $x > A^n$. It follows that

$$\frac{|x^{\frac{1}{n}} - s^{\frac{1}{n}}|}{|x - s|} = \frac{x^{\frac{1}{n}} - s^{\frac{1}{n}}}{x - s} \leq \frac{1}{nA^{n-1}}.$$

and thus

$$|x^{\frac{1}{n}} - s^{\frac{1}{n}}| \leq \frac{|x-s|}{nA^{n-1}},$$

for all real numbers x and s satisfying $s > A^n$ and $x > A^n$.

We claim that $\lim_{x\to s} x^{\frac{1}{n}} = s^{\frac{1}{n}}$. Indeed let some positive real number ε be given, and δ be a positive real number chosen small enough to ensure that both $s - \delta \geq A^n$ and $0 < \delta \leq nA^{n-1}\varepsilon$. If x is a real number satisfying $0 < |x - s| < \delta$ then $x > s - \delta \geq A^n$ and therefore

$$|x^{\frac{1}{n}} - s^{\frac{1}{n}}| \leq \frac{|x-s|}{nA^{n-1}} < \frac{\delta}{nA^{n-1}} \leq \frac{nA^{n-1}\varepsilon}{nA^{n-1}} = \varepsilon.$$

Thus $\lim_{x\to s} x^{\frac{1}{n}} = s^{\frac{1}{n}}$, as claimed.

Now the limit of a product of functions is the product of the limits of those functions (see Proposition 4.17). It follows that $\lim_{x\to s} x^{\frac{m}{n}} = s^{\frac{m}{n}}$ for all positive integers m and n.

Now earlier in the proof we showed that if m and n are positive integers, and if x and s are positive real numbers then

$$\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}} = \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} x^{\frac{j}{n}}.$$

Now the limit of a product of functions is the product of the limits of those functions, and the limit of a sum of functions is the sum of the limits of those functions (see Proposition 4.17). Applying these results, together with the Laws of Indices that apply to positive numbers raised to fractional powers (Proposition 1.14), we see that

$$\begin{aligned}
\lim_{x\to s}\left(\frac{x^{\frac{m}{n}} - s^{\frac{m}{n}}}{x^{\frac{1}{n}} - s^{\frac{1}{n}}}\right) &= \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} \lim_{x\to s} x^{\frac{j}{n}} \\
&= \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} \left(\lim_{x\to s} x^{\frac{1}{n}}\right)^j \\
&= \sum_{j=0}^{m-1} s^{\frac{m-1-j}{n}} s^{\frac{j}{n}} \\
&= ms^{\frac{m-1}{n}}.
\end{aligned}$$

Applying this result with $m = n$, we find that

$$\lim_{x\to s}\left(\frac{x^{\frac{1}{n}} - s^{\frac{1}{n}}}{x - s}\right) = \frac{1}{\lim_{x\to s}\left(\dfrac{x-s}{x^{\frac{1}{n}} - s^{\frac{1}{n}}}\right)} = \frac{1}{ns^{\frac{n-1}{n}}}.$$

It follows that

$$\begin{aligned}\lim_{x\to s}\left(\frac{x^{\frac{m}{n}}-s^{\frac{m}{n}}}{x-s}\right) &= \lim_{x\to s}\left(\frac{x^{\frac{m}{n}}-s^{\frac{m}{n}}}{x^{\frac{1}{n}}-s^{\frac{1}{n}}}\right)\times\lim_{x\to s}\left(\frac{x^{\frac{1}{n}}-s^{\frac{1}{n}}}{x-s}\right)\\ &= ms^{\frac{m-1}{n}}\times\frac{1}{ns^{\frac{n-1}{n}}}=\frac{m}{n}s^{\frac{m-n}{n}}=\frac{m}{n}s^{\frac{m}{n}-1},\end{aligned}$$

as required. ∎

Corollary 4.25 *Let q be a rational number. Then the function defined on the set of positive real numbers that maps each positive real number x to x^q is continuous and differentiable, and moreover*

$$\frac{d}{dx}\left(x^q\right)=qx^{q-1}.$$

Proof The result in the case $q > 0$ follows directly from Proposition 4.24. If $q = 0$ then the function f is constant, and the result is immediate.

Now suppose that $q < 0$. Let $p = -q$. Then

$$\frac{x^q-s^q}{x-s}=\frac{x^{-p}-s^{-p}}{x-s}=\frac{s^p-x^p}{x^ps^p(x-s)}=-\frac{x^p-s^p}{x^ps^p(x-s)}.$$

It follows from Proposition 4.24 and Proposition 4.17 that

$$\begin{aligned}\lim_{x\to s}\frac{f(x)-f(s)}{x-s} &= \lim_{x\to s}\frac{x^q-s^q}{x-s}\\ &= -\lim_{x\to s}\frac{1}{x^ps^p}\times\lim_{x\to s}\frac{x^p-s^p}{x-s}\\ &= -\frac{1}{s^{2p}}\times ps^{p-1}=-ps^{-p-1}=qs^{q-1}.\end{aligned}$$

The result follows. ∎

Remark Let q be a rational number, and let $f\colon(0,+\infty)\to\mathbb{R}$ be the function defined so that $f(x) = x^q$ for all positive real numbers x.

Suppose that $q > 0$. Then the function f is increasing and its range is the set $(+,\infty)$ of positive real numbers. The continuity of f at a positive real number s can therefore be shown as follows.

Let ε be a positive real number, and let δ be the minimum of the positive numbers $(s^q+\varepsilon)^{\frac{1}{q}}-s$ and $s-(s^q-\varepsilon)^{\frac{1}{q}}$. If $s-\delta<x<s+\delta$ then $s^q-\varepsilon<x^q<s^q+\varepsilon$. Thus the function f is continuous.

A similar argument shows that the function f is continuous in the case when $q < 0$. In this case the function is decreasing.

Proposition 4.26 *Let D and E be subsets of the set $\mathbb{R}$ of real numbers, let $f: \mathbb{R}$ and $g: E \to \mathbb{R}$ be continuous functions defined on D and E respectively, where $f(D) \subset E$, and let $g \circ f: D \to \mathbb{R}$ denote the composition of these functions, defined so that $(g \circ f)(x) = g(f(x))$ for all real numbers x belonging to D. Let s be a real number belonging to D. Suppose that the function f is continuous at s and that the function g is continuous at $f(s)$. Then the composition function $g \circ f$ is continuous at s.*

Proof Let some strictly positive real number ε be given. Then there exists some strictly positive real number η such that $|g(y) - g(f(s))| < \varepsilon$ for all real numbers y belonging to E that satisfy $|y - f(s)| < \eta$, because the function g is continuous at $f(x)$. But then there exists some strictly positive real number δ such that $|f(x) - f(s)| < \eta$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$. It follows that $|g(f(x)) - g(f(s))| < \varepsilon$ for all real numbers x belonging to D that satisfy $|x - s| < \delta$, and thus the function $g \circ f$ is continuous at s, as required. ∎

Example Let $f: \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = \sqrt{1 + 3x^2}$ for all real numbers x. The function $x \mapsto 1 + 3x^2$ is a polynomial function. It is therefore continuous on $\mathbb{R}$. (see Proposition 4.23). Also the function $u \mapsto \sqrt{u}$ is continuous on the set of positive real numbers (see Corollary 4.25). The function f is the composition of these two continuous functions. Therefore it is itself continuous. It then follows, for example, that

$$\lim_{x \to 1} \sqrt{1 + 3x^2} = \lim_{x \to 1} f(x) = f(1) = 2.$$

Example Let $f: \mathbb{R} \to \mathbb{R}$ be defined such that

$$f(x) = \frac{1 + \sqrt[5]{1 + x^2}}{\sqrt[3]{x^2 - 6x + 25}}$$

for all real numbers x. We show that this function is continuous.

Now $x^2 - 6x + 25 > 0$ for all real numbers x. Polynomial functions are continuous (see Proposition 4.23). It follows that the function $x \mapsto x^2 - 6x + 25$ is continuous on $\mathbb{R}$. The function $u \mapsto \sqrt[3]{u}$ is continuous on the set of positive real numbers (see Corollary 4.25). Thus the function

$$x \mapsto \sqrt[3]{x^2 - 6x + 25}$$

is a composition of two continuous functions, and is thus itself a continuous function on $\mathbb{R}$ (see Proposition 4.26).

Similarly the function $x \mapsto \sqrt[5]{1+x^2}$ is a continuous function on $\mathbb{R}$, and therefore the function

$$x \mapsto 1 + \sqrt[5]{1+x^2}$$

is a continuous function on $\mathbb{R}$. It follows that the function f is a quotient of two continuous functions. It is therefore itself a continuous function (see Proposition 4.22).

Proposition 4.27 *Let D and E be subsets of the set $\mathbb{R}$ of real numbers, let s be a limit point of D, let u be a point of E, let $f\colon D \to E$ be function satisfying $f(D) \subset E$, and let $g\colon E \to \mathbb{R}$ be a real-valued function on E. Suppose that*

$$\lim_{x \to s} f(x) = u$$

and that the function g is continuous at u. Then

$$\lim_{x \to s} g(f(x)) = g(u).$$

Proof Let some strictly positive real number ε be given. Then there exists some strictly positive real number η such that $|g(y) - g(u)| < \varepsilon$ for all real numbers y belonging to E that satisfy $|y - u| < \eta$, because the function g is continuous at u. But then there exists some positive real number δ such that $|f(x) - u| < \eta$ for all real numbers x belonging to D that satisfy $0 < |x - s| < \delta$. It follows that $|g(f(x)) - g(u)| < \varepsilon$ for all real numbers x belonging to D that satisfy $0 < |x - s| < \delta$, and thus

$$\lim_{x \to s} g(x) = g(u),$$

as required. ∎

Example We now show that the limit

$$\lim_{x \to 0} \sqrt[3]{\frac{54x^4 - 108x^5 + 60x^6}{2x^4 - 12x^6 + 24x^8}},$$

exists, and determine the value of this limit. Now if $x \neq 0$ then

$$\frac{54x^4 - 108x^5 + 60x^6}{2x^4 + 12x^6 + 3x^10} = \frac{54 - 108x + 60x^2}{2 + 12x^2 + 3x^6}.$$

Moreover

$$\lim_{x \to 0}(54 - 108x + 60x^2) = 54 \quad \text{and} \quad \lim_{x \to 0}(2 + 12x^2 + 3x^6) = 2.$$

It follows that

$$\begin{aligned}\lim_{x\to 0}\frac{54x^4-108x^5+60x^6}{2x^4+12x^6+3x^{10}} &= \lim_{x\to 0}\frac{54-108x+60x^2}{2+12x^2+3x^6}\\ &= \frac{\lim\limits_{x\to 0}(54-108x+60x^2)}{\lim\limits_{x\to 0}(2+12x^2+3x^6)}\\ &= \frac{54}{2}=27.\end{aligned}$$

The function defined on the set of positive real numbers that sends each positive real number u to $\sqrt[3]{u}$ is continuous. On applying Proposition 4.27, we see that

$$\lim_{x\to 0}\sqrt[3]{\frac{54x^4-108x^5+60x^6}{2x^4-12x^6+24x^8}}=\sqrt[3]{27}=3.$$

4.11 The Intermediate Value Theorem

The following theorem is stated *without proof.*

Theorem 4.28 (The Intermediate Value Theorem) *Let $f\colon[a,b]\to\mathbb{R}$ be a continuous real-valued function defined on a closed interval $[a,b]$, and let c be a real number that is between $f(a)$ and $f(b)$ (so that either $f(a)\le c\le f(b)$ or else $f(a)\ge c\ge f(b)$). Then there exists a real number s satisfying $a\le s\le b$ for which $f(s)=c$.*

4.12 The Extreme Value Theorem

The following theorem is also stated *without proof.*

Theorem 4.29 (The Extreme Value Theorem) *Let $f\colon[a,b]\to\mathbb{R}$ be a continuous real-valued function defined on a closed interval $[a,b]$. Then there exist real numbers u and v in the interval $[a,b]$ such that*

$$f(u)\le f(x)\le f(v)$$

for all real numbers x belonging to the interval $[a,b]$.

4.13 One-Sided Limits

Let $f\colon D\to\mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a real number. We say that $f(x)$ is defined *for all real numbers x satisfying $x>s$ that lie sufficiently close to s* if there exists some real number u satisfying $u > s$ such that $x \in D$ and thus $f(x)$ is defined for all real numbers x satisfying $s < x < u$.

Similarly we say that $f(x)$ is defined *for all real numbers x satisfying $x < s$ that lie sufficiently close to s* if there exists some real number u satisfying $u < s$ such that $x \in D$ and thus $f(x)$ is defined for all real numbers x satisfying $u < x < s$.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s and L be a real numbers. Suppose that $f(x)$ is defined for all real numbers x satisfying $x > s$ that lie sufficiently close to s. The real number L is said to be the *limit* of $f(x)$, as x tends to s from above if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that
>
> $$L - \varepsilon < f(x) < L + \varepsilon$$
>
> for all real numbers x that satisfy $s < x < s + \delta$.

In a situation where $f\colon D \to \mathbb{R}$ is a real-valued function, where s and L are a real numbers, and where L is the limit of $f(x)$ as x tends to s from above, then we can denote this fact by writing

$$\lim_{x \to s^+} f(x) = L.$$

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set of real numbers, and let s and L be real numbers. We suppose that $f(x)$ is defined for all real numbers x satisfying $x > s$ that lie sufficiently close to s. Then the real number L is the limit of $f(x)$ as x tends to s from above if and only if L is the limit of $f(x)$ as x tends to s in $D \cap [s, +\infty)$. Therefore all general results concerning limits can be applied to limits from above.

Thus let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions defined over a subset D of the set of real numbers, and let s be a real number. Suppose that $f(x)$ and $g(x)$ are defined for all real numbers x satisfying $x < s$ that lie sufficiently close to s. Suppose also that the limits

$$\lim_{x \to s^+} f(x) \quad \text{and} \quad \lim_{x \to s^+} g(x)$$

exist. It then follows from Proposition 4.17 that

$$\begin{aligned}
\lim_{x \to s^+} (f(x) + g(x)) &= \lim_{x \to s^+} f(x) + \lim_{x \to s^+} g(x), \\
\lim_{x \to s^+} (f(x) - g(x)) &= \lim_{x \to s^+} f(x) - \lim_{x \to s^+} g(x), \\
\lim_{x \to s^+} (f(x)g(x)) &= \lim_{x \to s^+} f(x) \times \lim_{x \to s^+} g(x),
\end{aligned}$$

Also it follows from Proposition 4.27 that

$$\lim_{x\to s^+} h(f(x)) = h\left(\lim_{x\to s^+} f(x)\right)$$

for all real-valued functions $h\colon E \to \mathbb{R}$ that are defined and continuous throughout some neighbourhood of $\lim\limits_{x\to s^+} f(x)$.

If moreover $g(x) \neq 0$ for all real numbers x satisfying $x > s$ that lie sufficiently close to s, and if $\lim\limits_{x\to s^+} g(x) \neq 0$ then

$$\lim_{x\to s^+} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to s^+} f(x)}{\lim\limits_{x\to s^+} g(x)}.$$

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s and L be a real numbers. Suppose that $f(x)$ is defined for all real numbers x satisfying $x < s$ that lie sufficiently close to s. The real number L is said to be the *limit* of $f(x)$, as x tends to s from below if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that
>
> $$L - \varepsilon < f(x) < L + \varepsilon$$
>
> for all real numbers x that satisfy $s - \delta < x < s$.

In a situation where $f\colon D \to \mathbb{R}$ is a real-valued function, where s and L are a real numbers, and where L is the limit of $f(x)$ as x tends to s from below, then we can denote this fact by writing

$$\lim_{x\to s^-} f(x) = L.$$

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set of real numbers, and let s and L be real numbers. We suppose that $f(x)$ is defined for all real numbers x satisfying $x < s$ that lie sufficiently close to s. Then the real number L is the limit of $f(x)$ as x tends to s from below if and only if L is the limit of $f(x)$ as x tends to s in $D \cap (-\infty, s]$. Therefore all general results concerning limits can be applied to limits from below.

Thus let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions defined over a subset D of the set of real numbers, and let s be a real number. Suppose that $f(x)$ and $g(x)$ are defined for all real numbers x satisfying $x < s$ that lie sufficiently close to s. Suppose also that the limits

$$\lim_{x \to s^-} f(x) \quad \text{and} \quad \lim_{x \to s^-} g(x)$$

exist. It then follows from Proposition 4.17 that

$$\begin{aligned}
\lim_{x \to s^-} (f(x) + g(x)) &= \lim_{x \to s^-} f(x) + \lim_{x \to s^-} g(x), \\
\lim_{x \to s^-} (f(x) - g(x)) &= \lim_{x \to s^-} f(x) - \lim_{x \to s^-} g(x), \\
\lim_{x \to s^-} (f(x)g(x)) &= \lim_{x \to s^-} f(x) \times \lim_{x \to s^-} g(x),
\end{aligned}$$

Also it follows from Proposition 4.27 that

$$\lim_{x \to s^-} h(f(x)) = h\left(\lim_{x \to s^-} f(x)\right)$$

for all real-valued functions $h\colon E \to \mathbb{R}$ that are defined and continuous throughout some neighbourhood of $\lim_{x \to s^-} f(x)$.

If moreover $g(x) \neq 0$ for all real numbers x satisfying $x > s$ that lie sufficiently close to s, and if $\lim_{x \to s^-} g(x) \neq 0$ then

$$\lim_{x \to s^-} \frac{f(x)}{g(x)} = \frac{\lim_{x \to s^-} f(x)}{\lim_{x \to s^-} g(x)}.$$

Proposition 4.30 *Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s and L be a real numbers. Suppose that $f(x)$ is defined for all real numbers x satisfying $x \neq s$ that lie sufficiently close to s. Then*

$$\lim_{x \to s} f(x) = L$$

if and only if

$$\lim_{x \to s^+} f(x) = L \quad \textit{and} \quad \lim_{x \to s^-} f(x) = L.$$

Proof Suppose that

$$\lim_{x \to s} f(x) = L.$$

Let some positive real number ε be given. Then there exists some positive number δ such that $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $0 < |x - s| <$

δ. But then $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $s < x < s + \delta$ and $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $s - \delta < x < s$. It follows that

$$\lim_{x \to s^+} f(x) = L \quad \text{and} \quad \lim_{x \to s^-} f(x) = L.$$

Conversely suppose that

$$\lim_{x \to s^+} f(x) = L \quad \text{and} \quad \lim_{x \to s^-} f(x) = L.$$

Let some positive real number ε be given. Then there exist positive real numbers δ_1 and δ_2 such that $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $s < x < s + \delta_1$ and $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $s - \delta_2 < x < s$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and $|f(x) - l| < \varepsilon$ for all real numbers x satisfying $0 < |x - s| < \delta$. Thus

$$\lim_{x \to s} f(x) = L.$$

This completes the proof. ∎

Corollary 4.31 *Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers. and let s be a real number. Suppose that $f(x)$ is defined for all real numbers x satisfying $x \neq s$ that lie sufficiently close to s. Then the function f is continuous at s if and only if*

$$\lim_{x \to s^+} f(x) = \lim_{x \to s^-} f(x) = f(s).$$

Proof The result follows immediately on combining the results of Proposition 4.21 Proposition 4.30. ∎

Example We determine the unique value of the constant c for which the function $f_c: \mathbb{R} \to \mathbb{R}$ is continuous, where

$$f_c(x) = \begin{cases} x^2 + x & \text{if } x \leq 2; \\ 5x + c & \text{if } x > 2. \end{cases}$$

We also investigate whether the resulting function is differentiable at $x = 2$.

Now

$$\lim_{x \to 2^-} f_c(x) = \lim_{x \to 2} (x^2 + x) = 2^2 + 2 = 6 = f_c(2)$$

and

$$\lim_{x \to 2^+} f_c(x) = \lim_{x \to 2} (5x + c) = 5 \times 2 + c = 10 + c.$$

It follows that

$$\lim_{x\to s^-} f_c(x) = \lim_{x\to s^+} f_c(x) = f_c(2)$$

if and only if $c = -4$.

We now investigate the differentiability at 2 of $f\colon \mathbb{R} \to \mathbb{R}$, where

$$f(x) = f_{-4}(x) = \begin{cases} x^2 + x & \text{if } x \leq 2; \\ 5x - 4 & \text{if } x > 2. \end{cases}$$

Now

$$\begin{aligned} \lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} &= \lim_{x\to 2^-} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2^-} \frac{(x+3)(x-2)}{x-2} \\ &= \lim_{x\to 2^-} (x+3) = 5, \end{aligned}$$

and

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{5x - 10}{x - 2} = 5.$$

Applying Proposition 4.30, we see that

$$\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = 5.$$

Thus the function f is differentiable at 2, and $f'(2) = 5$.

Note that, in this example

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} \quad \text{and} \quad \lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2}$$

are equal to the values at $x = 2$ of the derivatives of the functions $x \mapsto x^2 + x$ and $x \mapsto 5x - 4$ respectively.

4.14 Limits as the Variable Tends to Infinity

We now give the formal definition of the limit

$$\lim_{x\to +\infty} f(x)$$

of a real-valued function as the variable x "tends to $+\infty$".

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of R. We say that $f(x)$ is defined *for all sufficiently large values of* x if there exists a real number A with the property that $x \in D$ and thus $f(x)$ is defined for all real numbers x that satisfy $x > A$.

Note that, in the definitions and proofs that follow, all "positive" real numbers are strictly greater than zero. (The terms "positive" and "strictly positive" are synonymous: the word "strictly" may occasionally precede the word "positive" on occasion to emphasize the requirement that the quantity in question be strictly greater than zero.)

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers let $f\colon D \to \mathbb{R}$ be a real-valued function on D that is defined for all sufficiently large values of the real variable x. The real number L is said to be the *limit* of $f(x)$, as x tends to $+\infty$ if and only if the following criterion is satisfied:—

> given any positive real number ε, there exists some positive real number N such that
>
> $$L - \varepsilon < f(x) < L + \varepsilon$$
>
> for all real numbers x that satisfy $x > N$.

In a situation where $f\colon D \to \mathbb{R}$ is a real-valued function, where s and L are a real numbers, and where L is the limit of $f(x)$ as x tends to $+\infty$, then we can denote this fact by writing

$$\lim_{x \to +\infty} f(x) = L.$$

The following proposition is useful in enabling us to deduce immediately standard properties of limits of functions as the variable tends to infinity.

Proposition 4.32 *Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let L be a real number. Suppose that there exists a real number A large enough to ensure that $x \in D$ for all real numbers x satisfying $x > A$. Then*

$$\lim_{x \to +\infty} f(x) = L \quad \textit{if and only if} \quad \lim_{u \to 0^+} f\left(\frac{1}{u}\right) = L.$$

Proof Suppose that $\lim_{x \to +\infty} f(x) = L$. Let some positive real number ε be given. Let N be a positive real number, and let $\delta = \dfrac{1}{N}$. Then

$$L - \varepsilon < f(x) < L + \varepsilon$$

for all real numbers x satisfying $x > N$ if and only if

$$L - \varepsilon < f\left(\frac{1}{u}\right) < L + \varepsilon$$

for all real numbers u satisfying $0 < u < \delta$. The result follows. ∎

The following proposition follows on combining the results of Proposition 4.32, Proposition 4.17 and Proposition 4.27.

Proposition 4.33 *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be continuous functions defined over a subset D of the set of real numbers, and let s be a real number. Suppose that $f(x)$ and $g(x)$ are defined for all sufficiently large values of the real variable x. Suppose also that the limits*

$$\lim_{x\to+\infty} f(x) \quad \textit{and} \quad \lim_{x\to+\infty} g(x)$$

exist. Then

$$\begin{aligned}\lim_{x\to+\infty}(f(x)+g(x)) &= \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x),\\ \lim_{x\to+\infty}(f(x)-g(x)) &= \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x),\\ \lim_{x\to+\infty}(f(x)g(x)) &= \lim_{x\to+\infty} f(x) \times \lim_{x\to+\infty} g(x).\end{aligned}$$

Also

$$\lim_{x\to+\infty} h(f(x)) = h\left(\lim_{x\to+\infty} f(x)\right)$$

for all real-valued functions $h\colon E \to \mathbb{R}$ that are defined and continuous throughout some neighbourhood of $\lim_{x\to+\infty} f(x)$.

If moreover $g(x) \neq 0$ for all real numbers x satisfying $x > s$ that lie sufficiently close to s, and if $\lim_{x\to+\infty} g(x) \neq 0$ then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to+\infty} f(x)}{\lim\limits_{x\to+\infty} g(x)}.$$

Remark We have indicated how to apply the result of Proposition 4.32 in order to deduce the results stated in Proposition 4.33 from standard theorems that apply when a real variable approaches a limit point of its domain. The results of Proposition 4.33 can also be proved directly.

For example, suppose that

$$\lim_{x\to+\infty} f(x) = L_1 \quad \text{and} \quad \lim_{x\to+\infty} g(x) = L_2.$$

Let some positive real number ε be given. Then there exist positive real numbers N_1 and N_2 such that

$$L_1 - \tfrac{1}{2}\varepsilon < f(x) < L_1 + \tfrac{1}{2}\varepsilon$$

whenever $x > N_1$ and

$$L_2 - \tfrac{1}{2}\varepsilon < g(x) < L_2 + \tfrac{1}{2}\varepsilon$$

whenever $x > N_2$. Let N be the maximum of N_1 and N_2. If $x > N$ then

$$L_1 + L_2 - \varepsilon < f(x) + g(x) < L_1 + L_2 + \varepsilon$$

It follows that

$$\lim_{x\to+\infty} (f(x) + g(x)) = L_1 + L_2 = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x).$$

Example We show that

$$\lim_{x\to+\infty} \sqrt{\frac{16x^6 - 8x^3 + 5}{x^6 - 6x^5 + 15x^4}}$$

exists and determine its value. Now

$$\frac{16x^6 - 8x^3 + 5}{x^6 - 6x^5 + 15} = \frac{16 - 8x^{-3} + 5x^{-5}}{1 - 6x^{-1} + 15x^{-2}}$$

for all positive real numbers x. Moreover $\lim\limits_{x\to+\infty} x^{-1} = 0$, and therefore

$$\lim_{x\to+\infty} (16 - 8x^{-3} + 5x^{-5}) = 16 \quad \text{and} \quad \lim_{x\to+\infty} (1 - 6x^{-1} + 15x^{-2}) = 1.$$

It follows that

$$\begin{aligned} \lim_{x\to+\infty} \left(\frac{16x^6 - 8x^3 + 5}{x^6 - 6x^5 + 15} \right) &= \lim_{x\to+\infty} \left(\frac{16 - 8x^{-3} + 5x^{-5}}{1 - 6x^{-1} + 15x^{-2}} \right) \\ &= \frac{\lim\limits_{x\to+\infty} (16 - 8x^{-3} + 5x^{-5})}{\lim\limits_{x\to+\infty} (1 - 6x^{-1} + 15x^{-2})} \\ &= \frac{16}{1} = 16, \end{aligned}$$

and therefore

$$\lim_{x\to+\infty} \sqrt{\frac{16x^6 - 8x^3 + 5}{x^6 - 6x^5 + 15x^4}} = \sqrt{16} = 4.$$

We can define also the concept of the limit $\lim\limits_{x\to-\infty} f(x)$ of a real-valued function $f\colon D \to \mathbb{R}$ as x "tends to $-\infty$".

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers let $f\colon D \to \mathbb{R}$ be a real-valued function on D that is defined for all sufficiently large values of $-x$. The real number L is said to be the *limit* of $f(x)$, as x tends to $-\infty$ if and only if the following criterion is satisfied:—

> given any positive real number ε, there exists some positive real number N such that
> $$L - \varepsilon < f(x) < L + \varepsilon$$
> for all real numbers x that satisfy $x < -N$.

In a situation where $f\colon D \to \mathbb{R}$ is a real-valued function, where s and L are a real numbers, and where L is the limit of $f(x)$ as x tends to $-\infty$, then we can denote this fact by writing

$$\lim_{x \to -\infty} f(x) = L.$$

The relevant definitions ensure that

$$\lim_{x \to -\infty} f(x) = L \quad \text{if and only if} \quad \lim_{x \to +\infty} f(-x) = L.$$

Properties of limits as $x \to -\infty$ therefore follow directly from the properties of corresponding limits as $x \to +\infty$.

4.15 Functions Increasing and Decreasing without Bound

Definition let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a limit point of D. We say that $f(x)$ *increases without bound* as x tends to s, and write $f(x) \to +\infty$ as $x \to s$, if and only if the following criterion is satisfied:—

> given any positive real number M, there exists some positive real number δ such that
> $$f(x) > M$$
> for all real numbers x that satisfy $0 < |x - s| < \delta$.

Lemma 4.34 *let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a limit point of D. Suppose that $f(x) > 0$ for all $x \in D$. Then $f(x) \to +\infty$ as $x \to s$ if and only if*

$$\lim_{x \to s} \frac{1}{f(x)} = 0.$$

Proof Suppose that $f(x)$ increases without bound as $x \to s$. Let some positive real number ε be given. Then there exists some positive real number δ such that

$$f(x) > \frac{1}{\varepsilon}$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta$. But then

$$0 < \frac{1}{f(x)} < \varepsilon$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta$, and therefore

$$\lim_{x \to s} \frac{1}{f(x)} = 0.$$

Conversely suppose that

$$\lim_{x \to s} \frac{1}{f(x)} = 0,$$

where $f(x) > 0$ for all $x \in D$. Let some positive real number M be given. The formal definition of limits then ensures the existence of a positive real number δ such that

$$0 < \frac{1}{f(x)} < \frac{1}{M}$$

for all real numbers x in D that satisfy $0 < |x - s| < \delta$. But then $f(x) > M$ for all real numbers x in D that satisfy $0 < |x - s| < \delta$, and thus $f(x)$ increases without bound as $x \to s$. ∎

Definition let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a limit point of D. We say that $f(x)$ *decreases without bound* as x tends to s, and write $f(x) \to -\infty$ as $x \to s$, if and only if the following criterion is satisfied:—

> given any positive real number M, there exists some positive real number δ such that
>
> $$f(x) < -M$$
>
> for all real numbers x that satisfy $0 < |x - s| < \delta$.

The following lemma follows immediately from the formal definitions of what is meant by saying that $f(x) \to +\infty$ as $x \to s$ and $-f(x) \to -\infty$ as $x \to s$.

Lemma 4.35 *let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set of real numbers, and let s be a limit point of D. Then $f(x) \to +\infty$ as $x \to s$ if and only if $-f(x) \to -\infty$ as $x \to s$.*

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that $f(x)$ is defined for all real numbers x satisfying $x > s$ that lie sufficient close to s. We say that $f(x)$ *increases without bound* as x tends to s from above, and write $f(x) \to +\infty$ as $x \to s^+$ if and only if the following criterion is satisfied:—

> given any positive real number M, there exists some positive real number δ such that
> $$f(x) > M$$
> for all real numbers x that satisfy $s < x < s + \delta$.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that $f(x)$ is defined for all real numbers x satisfying $x < s$ that lie sufficient close to s. We say that $f(x)$ *increases without bound* as x tends to s from below, and write $f(x) \to +\infty$ as $x \to s^-$ if and only if the following criterion is satisfied:—

> given any positive real number M, there exists some positive real number δ such that
> $$f(x) > M$$
> for all real numbers x that satisfy $s - \delta < x < s$.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that there exists a constant A such that $f(x)$ is defined for all real numbers x satisfying $x > A$. We say that $f(x)$ *increases without bound* as x increases without bound, and write $f(x) \to +\infty$ as $x \to +\infty$ if and only if the following criterion is satisfied:—

> given any positive real number M, there exists a real number N such that $f(x) > M$ for all real numbers x that satisfy $x > N$.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that there exists a constant A such that $f(x)$ is defined for all real numbers x satisfying $x > A$. We say that $f(x)$ *decreases without bound* as x increases without bound, and write $f(x) \to -\infty$ as $x \to +\infty$ if and only if the following criterion is satisfied:—

given any positive real number M, there exists a positive real number N such that $f(x) < M$ for all real numbers x that satisfy $x > N$.

The following result follows directly on comparing the relevant definitions.

Proposition 4.36 *Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a real number. Suppose that there exists a constant A such that $f(x)$ is defined for all real numbers x satisfying $x > A$. Then the following statements are equivalent:—*

(i) $f(x) \to +\infty$ *as* $x \to +\infty$;

(ii) $-f(x) \to -\infty$ *as* $x \to +\infty$;

(iii) $f\left(\dfrac{1}{u}\right) \to +\infty$ *as* $u \to 0^+$;

(iv) $f(x) > 0$ *for all sufficiently large real numbers* x *and* $\displaystyle\lim_{x\to+\infty} \frac{1}{f(x)} = 0$;

(v) $f\left(\dfrac{1}{u}\right) > 0$ *for all sufficiently small positive real numbers* u *and* $\displaystyle\lim_{u\to 0^+} \frac{1}{f\left(\dfrac{1}{u}\right)} = 0$.

Example Let

$$f(x) = \sqrt[3]{\frac{x^3 + x}{8x^2 - 32x + 64}}$$

for all positive real numbers x. We consider the behavour of $f(x)$ as the variable x increases without bound. Now

$$\begin{aligned} f(x) &= \sqrt[3]{x}\,\sqrt[3]{\frac{x^2 + 1}{8x^2 - 32x^2 + 64}} \\ &= \sqrt[3]{x}\,\sqrt[3]{\frac{1 + x^{-2}}{8 - 32x^{-1} + 64x^{-2}}}. \end{aligned}$$

Moreover

$$\lim_{x\to+\infty} (1 + x^{-2}) = \lim_{u\to 0^+} (1 + u) = 1$$

and

$$\lim_{x\to+\infty} (8 + 32x^{-1} + 64x^{-2}) = \lim_{u\to 0^+} (8 + 32u + 64u^2) = 8.$$

The limit of a quotient of functions is the quotient of the limits, where those limits exist and the denominator is everywhere non-zero and has non-zero limit (see Proposition 4.33). It follows that

$$\lim_{x\to+\infty}\left(\frac{1+x^{-2}}{8-32x^{-1}+64x^{-2}}\right)=\frac{1}{8}.$$

It then follows from the continuity of the cube root function that

$$\lim_{x\to+\infty}\sqrt[3]{\frac{1+x^{-2}}{1-4x^{-1}+8x^{-2}}}=\frac{1}{2}.$$

Now

$$\lim_{x\to+\infty}\frac{1}{\sqrt[3]{x}}=\lim_{u\to0^+}\frac{1}{\sqrt[3]{u^{-1}}}=\lim_{u\to0^+}\sqrt[3]{u}=0.$$

It follows that

$$\begin{aligned}\lim_{x\to+\infty}\frac{1}{f(x)} &= \lim_{x\to+\infty}\frac{1}{\sqrt[3]{x}}\times\lim_{x\to+\infty}\frac{1}{\sqrt[3]{\dfrac{1+x^{-2}}{8-32x^{-1}+64x^{-2}}}}\\ &= 0.\end{aligned}$$

Moreover $f(x)>0$ for all positive numbers x. On applying property (iv) listed in the statement of Proposition 4.36, we conclude that that $f(x)\to+\infty$ as $x\to+\infty$.