Module MA1S11 (Calculus)
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Section 3: Functions

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Contents

3 Functions

3.1 Functions between Sets

Definition Let X and Y be sets. A *function* $f\colon X \to Y$ from X to Y assigns to each element x of the set X a corresponding element $f(x)$ of the set Y. The set X on which the function is defined is referred to as the *domain* of the function $f\colon X \to Y$. The set Y that contains the values of the function is referred to as the *codomain* of the function.

The following observations are important.

- The is no restriction on the nature of the contents of the sets X and Y appearing as the domain or codomain of a function. These sets could for example contain numbers, or words, or strings of characters representing DNA sequences, or colours, or students registered for a particular module.
- A function $f\colon X \to Y$ with domain X and codomain Y must assign a value $f(x)$ in Y to every single element x of the set X. Otherwise the definition of a function with domain X and codomain Y is not satisfied.
- In order to specify a function completely, it is necessary to specify both the domain X and the codomain Y of the function $f\colon X \to Y$.
- Algebraic formulae such as $\dfrac{x + 2x^3 + 3x^4}{\sqrt{1 - x^2}}$ may play a significant role in the specification of functions, but the concept of an algebraic expression of this sort is distinct from the concept of a function. Moreover a single algebraic expression of this sort is often not sufficient to specify some given mathematical function.

Example There is no function with domain $\mathbb{R}$ and codomain $\mathbb{R}$ that maps x to $1/x$ for all real numbers x. This is because $1/x$ is not a well-defined real number when $x = 0$. The reciprocal function is regarded as a function

$$r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

defined so that

$$r(x) = \frac{1}{x} \quad \text{for all non-zero real numbers } x.$$

The set $\mathbb{R} \setminus \{0\}$ may be regarded as the "natural domain" of the reciprocal function. It is obtained from the set $\mathbb{R}$ of real numbers by subtracting from it the singleton set $\{0\}$ consisting of just the real number 0. It follows that this set $\mathbb{R} \setminus \{0\}$ is the set consisting of all non-zero real numbers.

Example There is a well-defined function $f\colon [-1,1] \to \mathbb{R}$ defined so that

$$f(x) = \sqrt{1-x^2} \quad \text{for all } x \in [-1,1],$$

where, in accordance with standard notation for intervals,

$$[-1,1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}.$$

It should be noted that, in accordance with a standard convention in mathematics, and in cases where u is a non-negative real number, the symbol $\sqrt{u}$ denotes the unique *non-negative* real number satisfying $(\sqrt{u})^2 = u$.

Note that if $g\colon D \to \mathbb{R}$ is a function with domain D and codomain $\mathbb{R}$, where D is a subset of the set $\mathbb{R}$ of real numbers, and if the function g is defined so that $g(x) = \sqrt{1-x^2}$ for all $x \in D$, then it must be the case that $D \subset [-1,1]$. Thus $[-1,1]$ is the largest subset of the set $\mathbb{R}$ of the real numbers that can serve as the domain of a function mapping each real number x belonging to that domain to a corresponding real number $\sqrt{1-x^2}$. For this reason one may regard $[-1,1]$ as being the "natural domain" of a function f, mapping into the set of real numbers, that has the property that $f(x) = \sqrt{1-x^2}$ for all elements x of the domain of the function.

3.2 Injective Functions

Let X be a set, and let u and v be elements of X. We say that u and v are *distinct* if $u \neq v$.

Definition A function $f\colon X \to Y$ from a set X to a set Y is *injective* if it maps distinct elements of the set X to distinct elements of the set Y.

Let X and Y be sets, and let $f\colon X \to Y$ be a function from X to Y. Then $f\colon X \to Y$ is injective if and only if $f(u) \neq f(v)$ for all elements u and v of the domain X of the function that satisfy $u \neq v$.

One can show that a function $f\colon X \to Y$ is injective by proving that

$$u, v \in X \text{ and } f(u) = f(v) \Longrightarrow u = v.$$

(In other words, one can show that a function $f\colon X \to Y$ is injective by proving that, if u and v are elements of the domain of the function, and if $f(u) = f(v)$, then $u = v$.)

Example Let $f\colon [0,+\infty) \to \mathbb{R}$ be the function defined such that

$$f(x) = \frac{x}{x+1}$$

for all non-negative real numbers x. Let u and v be non-negative real numbers. Suppose that $f(u) = f(v)$. Then

$$\frac{u}{u+1} = \frac{v}{v+1}.$$

Multiplying both sides of this equation by $(u+1)(v+1)$, we find that

$$u(v+1) = v(u+1).$$

and thus

$$uv + u = uv + v.$$

Subtracting uv from both sides of this equation, we find that $u = v$. This proves that the function $f\colon [0, +\infty) \to \mathbb{R}$ is injective.

Lemma 3.1 *Let X and Y be sets, and let $f\colon X \to Y$ be a function from X to Y. The function f is injective if and only if, given any element y of the codomain Y, there exists at most one element x of the domain X satisfying $f(x) = y$.*

Proof The result follows directly from the definitions. Indeed if the function is injective, then there cannot exist distinct elements of X that map to the same element of Y, and thus each element of Y is the image of at most one element of X. Conversely if each element of Y is the image of at most one element of X then distinct elements of X cannot map to the same element of Y, and therefore distinct elements of X must map to distinct elements of Y, and thus the function $f\colon X \to Y$ is injective.

3.3 The Range of a Function

Definition Let X and Y be sets, and let $f\colon X \to Y$ be a function from X to Y. The *range* of $f\colon X \to Y$, often denoted by $f(X)$, consists of those elements of the codomain Y that are of the form $f(x)$ for at least one element x of X.

Let $f\colon X \to Y$ be a function from a set X to a set Y, and let $f(X)$ be the range of this function. Then

$$\begin{aligned} f(X) &= \{y \in Y \mid y = f(x) \text{ for some } x \in X\} \\ &= \{f(x) \mid x \in X\}. \end{aligned}$$

(We use above a standard notation from set theory: the specification $\{f(x) \mid x \in X\}$ denotes the set consisting of all objects that are of the form $f(x)$ for some element x of the set X.)

3.4 Surjective Functions

Definition Let X and Y be sets. A function $f: X \to Y$ is said to be *surjective* if, given any element y of the codomain Y, there exists at least one element x of the domain X that satisfies $f(x) = y$.

The following result follows directly from the relevant definitions.

Lemma 3.2 *A function $f: X \to Y$ from a set X to a set Y is surjective if and only if $f(X) = Y$, where $f(X)$ is the range of the function f, and Y is the codomain of the function.*

Example Consider the function $f: \mathbb{Z} \to \mathbb{Z}$ from the set $\mathbb{Z}$ of integers to itself defined such that $f(x) = 5x - 3$ for all $x \in \mathbb{Z}$. If u and v are integers, and if $f(u) = f(v)$ then $5u - 3 = 5v - 3$. Adding 3 to both sides of this equation, and then dividing by 5, we find that $5u = 5v$, and therefore $u = v$. We conclude that the function $f: \mathbb{Z} \to \mathbb{Z}$ is injective. Thus every integer k satisfies $f(n) = k$ for at most one integer n.

Now an integer k is of the form $f(n)$ for some integer n if and only if $k+3$ is divisible by 5. It follows that the range $f(\mathbb{Z})$ of the function f consists of those positive integers whose decimal representation has least significant digit 2 or 7, together with those negative integers whose decimal representation has least significant digit 8 or 3. The range of the function is not equal to the codomain of the function, and therefore the function is not surjective.

Example Consider the function $p: \mathbb{Z} \to \mathbb{Z}$ defined such that

$$p(n) = \begin{cases} n + 3 & \text{if } n \text{ is an odd integer;} \\ n - 5 & \text{if } n \text{ is an even integer.} \end{cases}$$

First we note that $p(n)$ is odd for all even integers n, and $p(n)$ is even for all odd integers n. We determine whether or not the function p is injective, and whether or not it is surjective.

Let m and n be integers for which $p(m) = p(n)$. Suppose first that $p(m)$ is even. Then m and n are odd, and

$$m = p(m) - 3 = p(n) - 3 = n.$$

Next suppose that $p(m)$ is odd. Then m and n are even, and

$$m = p(m) + 5 = p(n) + 5 = n.$$

Now $p(m)$ is either even or odd. It follows that $m = n$ whenever $p(m) = p(n)$. Thus the function p is injective.

Let k be an integer. If k is even then $k = p(k-3)$. If k is odd then $k = p(k+5)$. Every integer k is even or odd. It follows that every integer is in the range of the function p, and thus this function is surjective.

We have thus shown that the function $p\colon \mathbb{Z} \to \mathbb{Z}$ is both injective and surjective.

3.5 Bijective Functions

Definition A function $f\colon X \to Y$ from a set X to a set Y is said to be *bijective* if it is both injective and surjective.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself. defined such that $f(x) = 5x - 3$ for all real numbers x. The function f is injective. Indeed if u and v are real numbers, and if $f(u) = f(v)$ then $5u - 3 = 5v - 3$. But then $5u = 5v$, and therefore $u = v$.

Now $y = f(\frac{1}{5}(y+3))$ for all real numbers y. It follows that every element of the codomain $\mathbb{R}$ of the function is in the range of the function. This demonstrates that the function $f\colon X \to Y$ is surjective (see Lemma 3.2).

The function $f\colon \mathbb{R} \to \mathbb{R}$ has been shown to be both injective and surjective. It is therefore bijective.

3.6 Inverses of Functions

Definition Let X and Y be sets, and let $f\colon X \to Y$ be a function from X to Y. A function $g\colon Y \to X$ is said to be the *inverse* of $f\colon X \to Y$ if $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be the functions defined such that $f(x) = 5x-3$ for all real numbers x and $g(y) = \frac{1}{5}(y+3)$ for all real numbers y. Then

$$g(f(x)) = \tfrac{1}{5}(f(x) + 3) = \tfrac{1}{5}(5x) = x$$

for all real numbers x, and

$$f(g(y)) = 5 \times \tfrac{1}{5}(y+3) - 3 = (y+3) - 3 = y$$

for all real numbers y. It follows that the function $g\colon \mathbb{R} \to \mathbb{R}$ is the inverse of the function $f\colon \mathbb{R} \to \mathbb{R}$.

Example Let $p\colon \mathbb{Z} \to \mathbb{Z}$ be the function from the set $\mathbb{Z}$ of integers to itself defined such that

$$p(n) = \begin{cases} n+3 & \text{if } n \text{ is an odd integer;} \\ n-5 & \text{if } n \text{ is an even integer.} \end{cases}$$

We have already shown in an earlier example that this function is both injective and surjective. It is therefore bijective.

Let $q: \mathbb{Z} \to \mathbb{Z}$ be the function from the set $\mathbb{Z}$ of integers to itself defined such that

$$q(m) = \begin{cases} m + 5 & \text{if } m \text{ is an odd integer;} \\ m - 3 & \text{if } m \text{ is an even integer.} \end{cases}$$

If n is an odd integer then $p(n)$ is an even integer, and $p(n) = n + 3$, and therefore $q(p(n)) = p(n) - 3 = n$. If n is an even integer then $p(n)$ is an odd integer and $p(n) = n - 5$, and therefore $q(p(n)) = p(n) + 5 = n$. It follows that $p(n) = n$ for all integers n, irrespective of whether n is even or odd.

A similar argument shows that $p(q(m)) = m$ for all integers m. It follows that the function $q: \mathbb{Z} \to \mathbb{Z}$ is the inverse of the function $p: \mathbb{Z} \to \mathbb{Z}$.

Proposition 3.3 *Let X and Y be sets. A function $f: X \to Y$ from X to Y has a well-defined inverse $g: Y \to X$ if and only if $f: X \to Y$ is bijective.*

Proof First suppose that $f: X \to Y$ has a well-defined inverse $g: X \to Y$. We show that the function $f: X \to Y$ is then both injective and surjective.

Let u and v be elements of the domain X of f. Suppose that $f(u) = f(v)$. Then

$$u = g(f(u)) = g(f(v)) = v.$$

It follows that the function $f: X \to Y$ is injective. Also $y = f(g(y))$ for all elements y of the codomain Y of f, and therefore the function $f: X \to Y$ is surjective. We have thus proved that if $f: X \to Y$ has a well-defined inverse, then this function is bijective.

Now let $f: X \to Y$ be a function that is bijective. We must show that this function has a well-defined inverse $g: Y \to X$. Now the function $f: X \to Y$ is surjective, and therefore, given any element y of the codomain Y of f, there exists at least one element x of the domain X of f satisfying $f(x) = y$, because the function f is surjective. Also the function $f: X \to Y$ is injective, and therefore, given any element y of the codomain Y of f, there exists at most one element x of the domain X of f satisfying $f(x) = y$ (see Lemma 3.1). Putting these results together, we see that, given any element y of the codomain Y of f, there exists exactly one element x of the domain X of f satisfying $f(x) = y$. There therefore exists a function $g: Y \to X$ defined such that, given any element y of Y, $g(y)$ is the unique element of the set X that satisfies $f(g(y)) = y$.

Now the very definition of the function g ensures that $f(g(y)) = y$ for all $y \in Y$. We must also show that $g(f(x)) = x$ for all $x \in X$. Now, given $x \in X$, the element $g(f(x))$ is by definition the unique element u of the set X

satisfying $f(u) = f(x)$. But the injectivity of the function f ensures that if $u \in X$ satisfies $f(u) = f(x)$ then $u = x$. It follows that $g(f(x)) = x$ for all elements x of the domain X of f. We have now verified that the bijective function $f: X \to Y$ does indeed have a well-defined inverse $g: Y \to X$. This completes the proof. $\blacksquare$

3.7 Natural Domains

Example We consider what is the natural domain of a real-valued function f, where

$$f(x) = \sqrt{\frac{8}{\sqrt{x-1}+1} - 2}$$

for all elements x of this natural domain. (For the purposes of this example, we adopt the requirement that the square root $\sqrt{u}$ of a number is defined only when that number u is both real and non-negative.)

Now the inner square root needs to be defined. We therefore require that $x \geq 1$ for all elements x of the sought natural domain. Note that $\sqrt{x-1} + 1 \geq 1$ whenever $x \geq 1$, and therefore

$$\frac{8}{\sqrt{x-1}+1}$$

is defined for all real numbers x satisfying $x \geq 1$.

But it is also necessary to ensure that

$$\frac{8}{\sqrt{x-1}+1} - 2 \geq 0$$

for all elements x of the natural domain of the function. We therefore require that

$$\frac{8}{\sqrt{x-1}+1} \geq 2.$$

Now

$$\begin{aligned} & \frac{8}{\sqrt{x-1}+1} \geq 2. \\ \iff \quad & \sqrt{x-1}+1 \leq 4 \\ \iff \quad & \sqrt{x-1} \leq 3 \\ \iff \quad & 0 \leq x-1 \leq 9 \\ \iff \quad & 1 \leq x \leq 10 \end{aligned}$$

(Here the symbol $\iff$ means "if and only if".) It follows that the expression specifying the value of $f(x)$ is well-defined only if $1 \leq x \leq 10$. Moreover this expression does yield a well-defined real number for all real numbers x satisfying $1 \leq x \leq 10$. The natural domain of a function determined by the given expression is thus $[1, 10]$, where

$$[1, 10] = \{x \in \mathbb{R} \mid 1 \leq x \leq 10\}.$$

Thus the real-valued function with the most extensive domain specified by the given expression is the function

$$f \colon [1, 10] \to \mathbb{R}$$

defined such that

$$f(x) = \sqrt{\frac{8}{\sqrt{x-1}+1} - 2}$$

for all $x \in [1, 10]$. One can say that $[1, 10]$ is the "natural domain" for a function determined by the given expression.

3.8 Increase and Decrease of Functions of a Real Variable

Definition A function is said to be *real-valued* if its codomain is the set $\mathbb{R}$ of real numbers, or if its codomain is some subset of $\mathbb{R}$.

We consider the increase and decrease of real-valued functions whose domain is a subset of the set $\mathbb{R}$ of the real numbers, discussing intervals in the domain where such functions increase and decrease, and points in the domain at which such functions attain local minima and maxima.

Definition Let $f \colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of the set $\mathbb{R}$ of real numbers. Then:

the function $f \colon D \to \mathbb{R}$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all elements u and v of D satisfying $u \leq v$;

the function $f \colon D \to \mathbb{R}$ is said to be (strictly) *increasing* if $f(u) < f(v)$ for all elements u and v of D satisfying $u < v$;

the function $f \colon D \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all elements u and v of D satisfying $u \leq v$;

the function $f: D \to \mathbb{R}$ is said to be (strictly) *decreasing* if $f(u) > f(v)$ for all elements u and v of D satisfying $u < v$.

Definition A real-valued function $f: D \to \mathbb{R}$ defined over a subset D of the set $\mathbb{R}$ of real numbers is said to be *monotonic* if it is non-decreasing or non-increasing on D.

Lemma 3.4 *Let $f: D \to \mathbb{R}$ be a (strictly) increasing function defined over a subset D of the set $\mathbb{R}$ of real numbers. Then the function $f: D \to \mathbb{R}$ is injective.*

Proof Let u and v be distinct elements of the set D. Then either $u < v$ or $v < u$. If $u < v$ then $f(u) < f(v)$, because the function f is increasing, and therefore $f(u) \neq f(v)$. If $v < u$ then $f(v) < f(u)$ and therefore $f(u) \neq f(v)$. The result follows. ∎

Lemma 3.5 *Let $f: D \to \mathbb{R}$ be a (strictly) decreasing function defined over a subset D of the set $\mathbb{R}$ of real numbers. Then the function $f: D \to \mathbb{R}$ is injective.*

Proof Let u and v be distinct elements of the set D. Then either $u < v$ or $v < u$. If $u < v$ then $f(u) > f(v)$, because the function f is decreasing, and therefore $f(u) \neq f(v)$. If $v < u$ then $f(v) > f(u)$ and therefore $f(u) \neq f(v)$. The result follows. ∎

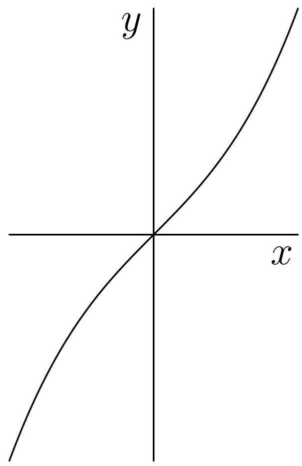
Example Let $f: \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself defined such that $f(x) = x^3 + x$ for all real numbers x.

Let u and v be real numbers satisfying $u < v$. If $u \geq 0$ and $v \geq 0$ then $u^3 < v^3$. If $u < 0$ is negative and $v \geq 0$ is non-negative then $u^3 < 0$ and $v^3 \geq 0$ and therefore $u^3 < v^3$. If $u < 0$ and $v < 0$ then $u = -|u|$, $v = -|v|$. Moreover $|u| > |v|$, because $u < v$. It follows that

$$u^3 = (-|u|)^3 = -|u|^3 < -|v|^3 = (-|v|)^3 = v^3.$$

The case when $v < 0$ and $u \geq 0$ does not arise, because $u < v$. We have therefore investigated all relevant cases determined by the signs of the real numbers u and v, and, in all cases, we have shown that $u^3 < v^3$. Thus $u^3 < v^3$ for all real numbers u and v satisfying $u < v$.

Adding the inequalities $u^3 < v^3$ and $u < v$ we find that $u^3 + u < v^3 + v$ whenever $u < v$. Thus $f(u) < f(v)$ for all real numbers u and v satisfying $u < v$. It follows that the function $f: \mathbb{R} \to \mathbb{R}$ is increasing, where $f(x) = x^3 + x$ for all real numbers x. It follows from Lemma 3.4 that this function is injective.



Graph of the curve $y = x^3 + x$.

Example Let $g\colon \mathbb{R} \to \mathbb{R}$ be the function from the set $\mathbb{R}$ of real numbers to itself defined such that $g(x) = x^3 - x$ for all real numbers x. Then $g(-1) = g(0) = g(1)$. It follows that the function g is not injective.

Example Let $a = \dfrac{1}{\sqrt{3}}$, let $h\colon [a, +\infty) \to \mathbb{R}$ be the function from $[a, +\infty)$ to $\mathbb{R}$ defined such that $h(x) = x^3 - x$ for all real numbers x satisfying $x \geq a$, and let $k\colon [-a, a] \to \mathbb{R}$ be the function defined such that $k(x) = x^3 - x$ for all real numbers x satisfying $-a \leq x \leq a$.

Let u and v be real numbers satisfying $a \leq u < v$. Then

$$v^3 - u^3 = (v - u)(v^2 + uv + u^2),$$

and therefore

$$h(v) - h(u) = (v - u)(v^2 + uv + u^2 - 1).$$

But $a \leq u < v$, and therefore

$$v^2 + uv + u^2 - 1 \geq 3a^2 - 1 = 0.$$

It follows that $h(v) > h(u)$ for all real numbers u and v satisfying $a \leq u < v$. Thus the function $h\colon [a, +\infty) \to \mathbb{R}$ is increasing on $[a, +\infty)$, where $a = \frac{1}{\sqrt{3}}$, and therefore this function h is injective.

Next let u and v be real numbers satisfying $-a \leq u < v \leq a$. Then

$$k(v) - k(u) = (v - u)(v^2 + uv + u^2 - 1).$$

If $u \geq 0$ and $v \geq 0$ then

$$v^2 + uv + u^2 - 1 < 3a^2 - 1 = 0$$

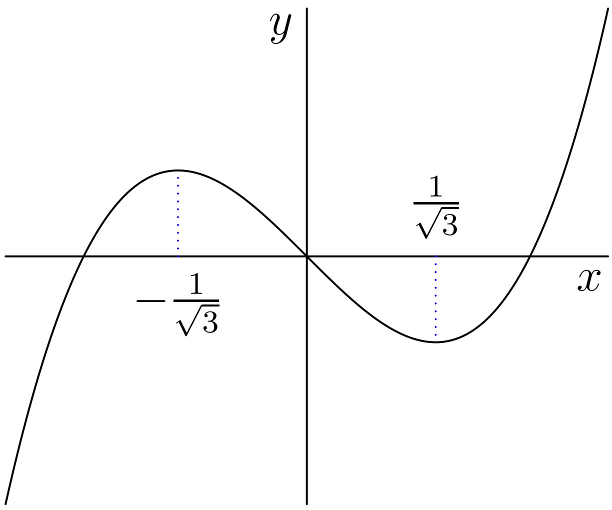
If $u < 0$ and $v < 0$ then

$$v^2 + uv + u^2 - 1 = (-|v|)^2 + (-|u|)(-|v|) + (-|u|)^2 - 1 < 3a^2 - 1 = 0,$$

and if $u < 0$ and $v \geq 0$ then $v^2 + uv + u^2 - 1 \leq -1$. It follows that

$$k(v) - k(u) = (v - u)(v^2 + uv + u^2 - 1) < 0$$

for all real numbers u and v satisfying $-a \leq u < v \leq a$. Thus the function $k: [-a, a] \to \mathbb{R}$ is decreasing on $[-a, a]$, where $a = \frac{1}{\sqrt{3}}$, and therefore this function k is injective.



Graph of the curve $y = x^3 - x$.

We now resume discussion of the function $g: \mathbb{R} \to \mathbb{R}$, where $g(x) = x^3 - x$ for all real numbers x. Let $a = \dfrac{1}{\sqrt{3}}$. We have proved that the function h obtained by restricting the function g to the interval $[a, +\infty)$ is an increasing function. It follows that $g(x) > g(a)$ whenever $x > a$. We have also proved that the function k obtained by restricting the function g to the interval $[-a, a]$ is a decreasing function. It follows that $g(x) > g(a)$ whenever $-a \leq x < a$. We conclude that $g(x) \geq g(a)$ for all real numbers x satisfying $x \geq -a$. This justifies the assertion that $g(x)$ attains a *local minimum* when $x = \frac{1}{\sqrt{3}}$.

We now obtain the corresponding result that $g(x)$ attains a *local maximum* when $x = -a$. Now the function $g\colon \mathbb{R} \to \mathbb{R}$ is an *odd* function. This means that $g(-x) = -g(x)$ for all real numbers x. It follows that $g(x) = -h(|x|)$ for all real numbers x satisfying $x < -a$, where $h\colon [a, +\infty) \to \mathbb{R}$ is defined such that $h(x) = x^3 - x$ for all real numbers x satisfying $x \geq a$. Now we have shown that the function h is an increasing function on the interval $[a, +\infty)$. It follows that if u and v are real numbers satisfying $u < v \leq -a$ then $a \leq |v| < |u|$. But then $h(|u|) > h(|v|)$, because the function h is increasing on $[a, +\infty)$. It follows that

$$g(u) = -h(|u|) < -h(|v|) = g(v).$$

We conclude from this that the function g is increasing on the interval $(-\infty, -a]$, where $a = \frac{1}{\sqrt{3}}$. We have already shown that the function g is decreasing on the interval $[-a, a]$ (because it is equal on this interval to the function $k\colon [-a, a] \to \mathbb{R}$, and we have shown that the function k is decreasing). It follows that $g(x) < g(-a)$ when $x < -a$ and $g(x) < g(-a)$ when $-a < x < a$. We conclude from this that $g(x) \leq g(-a)$ for all real numbers x satisfying $x \leq a$. This justifies the assertion that $g(x)$ attains a *local maximum* when $x = -\frac{1}{\sqrt{3}}$.

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set $\mathbb{R}$ of real numbers. We say that the function f attains a *local minimum* at an element s of D if there exists some positive real number δ such that $f(x) \geq f(s)$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$.

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set $\mathbb{R}$ of real numbers. We say that the function f attains a *local maximum* at an element s of D if there exists some positive real number δ such that $f(x) \leq f(s)$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$.

We now introduce the concept of a *neighbourhood* of a real number s in some subset D of $\mathbb{R}$ to which the real number s belongs.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let the real number s be an element of D. A subset N of D is said to be a *neighbourhood* of s (in D) if there exists some positive real number δ such that $x \in N$ for all real numbers x for which both $s - \delta < x < s + \delta$ and $x \in D$.

The formal definition of *neighbourhood* captures the notion that a subset N of D is a neighbourhood of the real number s if and only if N contains all elements of D that lie "sufficiently close" to s.

Another perhaps useful way of thinking about neighbourhoods is to observe that a subset N of a subset D of the set of real numbers is a neighbourhood of some element s of D if and only if N "completely surrounds" s in D, so that s cannot be "approached" within D without entering into the neighbourhood of s. (In the same way, one cannot approach the house of a friend without passing through a neighbourhood within which the house is located.)

The definitions of local maxima and minima may now be reformulated in perhaps more attractive terms as presented in the following lemmas, which follow directly from the relevant definitions.

Lemma 3.6 *Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set $\mathbb{R}$ of real numbers. The function f attains a* local minimum *at an element s of D if and only if there exists some neighbourhood N of s in D small enough to ensure that $f(x) \geq f(s)$ for all $x \in N$.*

Lemma 3.7 *Let $f: D \to \mathbb{R}$ be a real-valued function defined on a subset D of the set $\mathbb{R}$ of real numbers. The function f attains a* local maximum *at an element s of D if and only if there exists some neighbourhood N of s in D small enough to ensure that $f(x) \leq f(s)$ for all $x \in N$.*

Example Let a and c be real constants, where $a > 0$ and $c > 0$, and let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function defined on the set $\mathbb{R} \setminus \{0\}$ of non-zero real numbers so that

$$f(x) = ax + \frac{c}{x}$$

for all non-zero real numbers x. We shall investigate the qualitative behaviour of this function, and will in particular determine the range of the function f.

Let y be a real number belonging to the range of the function f, and let x be a non-zero real number satisfying $f(x) = y$. Then

$$y = ax + \frac{c}{x}.$$

If we multiply both sides of this identity by x, and then subtract the left hand side from the right hand side, we arrive at the equation

$$ax^2 - yx + c = 0.$$

The quadratic polynomial on the left hand side of this equation must have real roots if y is to belong to the range of the function f. It follows from the standard quadratic formula that y must satisfy the inequality $y^2 \geq 4ac$.

Conversely if the real number y satisfies the inequality $y^2 \geq 4ac$ then the polynomial has real roots, and y therefore belongs to the range of the function f. In particular, if $y^2 = 4ac$ then there is a unique non-zero real number x for which $f(x) = y$. Moreover the unique real number x_0 for which $f(x_0) = 2\sqrt{ac}$, is given by the formula

$$x_0 = \frac{y}{2a} = \frac{2\sqrt{ac}}{2a} = \sqrt{\frac{c}{a}},$$

and the unique real number x for which $f(x) = -2\sqrt{ac}$, is

$$x = \frac{y}{2a} = -\frac{2\sqrt{ac}}{2a} = -\sqrt{\frac{c}{a}} = -x_0.$$

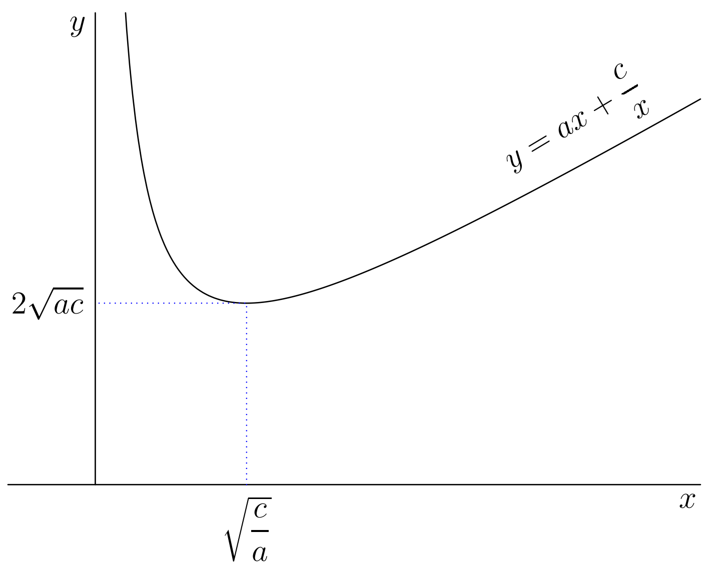
Next suppose that either $y > 2\sqrt{ac}$ or $y < -2\sqrt{ac}$. Then there exist two distinct non-zero real numbers x satisfying $f(x) = y$. They are

$$x = \frac{y \pm \sqrt{y^2 - 4ac}}{2a}.$$

We see from this that the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is not injective. The range of this function is the union

$$(-\infty, -2\sqrt{ac}] \cup [2\sqrt{ac}, +\infty).$$

It is now clear that the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is is not surjective.



Now it is clear that there exists a positive real number δ_0 such that $f(x) \geq f(x_0) = 2\sqrt{ac}$ whenever $x_0 - \delta_0 < x < x_0 + \delta_0$, where

$$x_0 = \sqrt{\frac{c}{a}}.$$

Indeed it suffices to ensure that all non-zero real numbers satisfying these inequalities are positive, and thus we may pick $\delta_0 = x_0$, or alternatively we may set δ_0 equal to any positive real number not exceeding x_0. In these circumstances we say that $f(x) \geq f(x_0)$ throughout some *neighbourhood* of x_0, and accordingly we say that the function f has a local minimum at x_0, where $x_0 = \sqrt{c/a}$.

Similarly we say that the function f has a *local maximum* at $-x_0$, where $x_0 = \sqrt{c/a}$.

We now show formally that the function f is increasing on the interval $[x_0, +\infty)$, where $x_0 = \sqrt{c/a}$. Let u and v be real numbers satisfying $x_0 \leq u < v$. Then

$$\begin{aligned} f(v) - f(u) &= av + \frac{c}{v} - au - \frac{c}{u} \\ &= \left(a - \frac{c}{uv}\right)(v - u). \end{aligned}$$

Now

$$v > u \geq x_0 = \sqrt{\frac{c}{a}},$$

and therefore

$$\frac{c}{uv} < \frac{c}{u^2} \leq \frac{c}{x_0^2} = a.$$

It follows that

$$\frac{f(v) - f(u)}{v - u} = a - \frac{c}{uv} > 0$$

whenever $x_0 \leq u < v$, and therefore the function f is increasing on the interval $[x_0, +\infty)$. Moreover, interchanging the roles of u and v, we find that the identity

$$\frac{f(v) - f(u)}{v - u} = a - \frac{c}{uv}$$

is valid for all real numbers u and v satisfying $u \geq x_0$, $v \geq x_0$ and $u \neq v$, irrespective of whether $u < v$ or $v < u$.

We now consider the behaviour of the function f on the interval $(0, x_0]$, where $x_0 = \sqrt{c/a}$. Let u and v be real numbers satisfying $u < v \leq x_0$. Then

$$f(v) - f(u) = \left(a - \frac{c}{uv}\right)(v - u).$$

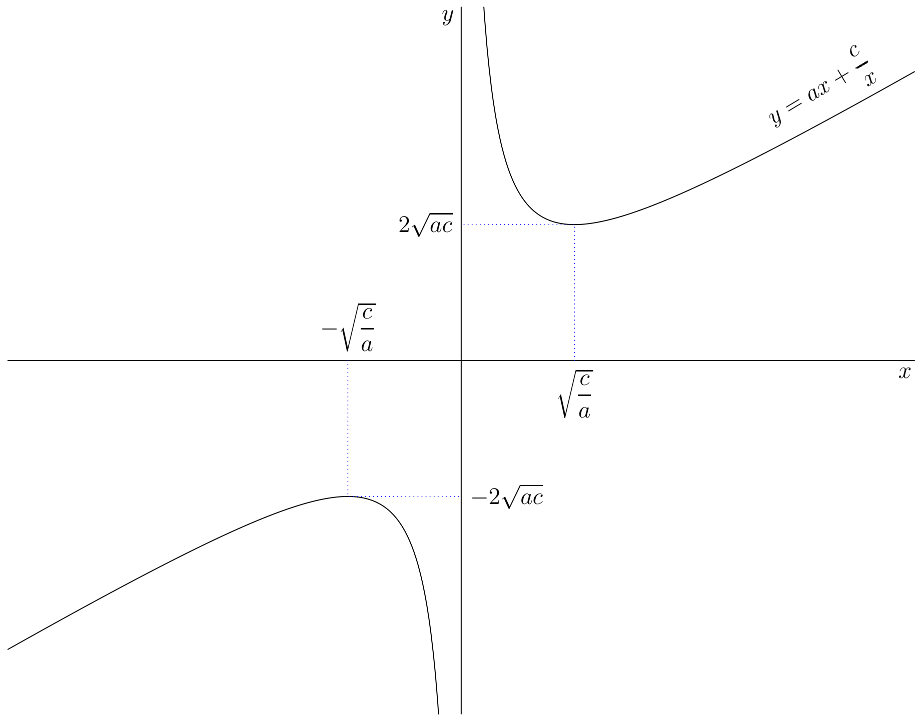
But now

$$\frac{c}{uv} > \frac{c}{v^2} \geq \frac{c}{x_0^2} = c \times \frac{a}{c} = a,$$

and therefore

$$a - \frac{c}{uv} < 0.$$

it follows that $f(v) < f(u)$ for all real numbers u and v satisfying $u < v < x_0$, and thus the function f is decreasing on the interval $(0, x_0]$, where $x_0 = \sqrt{c/a}$.



The behaviour of $f(x)$ when $x < 0$ may be determined from the results already obtained in view of the fact that the function f is an *odd function* which satisfies the identity $f(-x) = -f(x)$ for all non-zero real numbers x. If u and v are real numbers satisfying $u < v \leq -x_0 < 0$, where $x_0 = \sqrt{c/a}$, then $|u| > |v| \geq x_0$, and therefore

$$f(v) - f(u) = -f(|v|) + f(|u|) > 0.$$

Thus the function f is increasing on the interval $(-\infty, -x_0]$. Similarly if u and v are real numbers satisfying $-x_0 \leq u < v < 0$ then $0 \leq |v| < |u| \leq x_0$, and therefore

$$f(v) - f(u) = -f(|v|) + f(|u|) < 0.$$

Thus the function f is decreasing on the interval $[-x_0, 0)$, where $x_0 = \sqrt{c/a}$.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined such that

$$f(x) = x^3 - 9x^2 + 24x - 16$$

for all real numbers x. The cubic polynomial determining this function has an integer root equal to 1 because the sum of the coefficients of this polynomial are equal to zero. We identify the other roots of this polynomial and investigative the qualitative behaviour of the function.

The cubic polynomial may be divided by $x - 1$ using the standard polynomial division method:—

$$\begin{array}{rrrrr}
 & x^2 & -\,8x & +\,16 & \\
\cline{2-5}
x-1 \,\big) & x^3 \; -\,9x^2 & +\,24x & -\,16 & \\
 & x^3 \; -\,x^2 & & & \\
\cline{2-2}
 & -8x^2 & +\,24x & & \\
 & -8x^2 & +\,8x & & \\
\cline{2-3}
 & & 16x & -\,16 & \\
 & & 16x & -\,16 & \\
\cline{3-4}
 & & & 0 &
\end{array}$$

This calculation yields the result that

$$x^3 - 9x^2 + 24x - 16 = (x - 1)(x^2 - 8x + 16).$$

Factorizing the quadratic, we find that

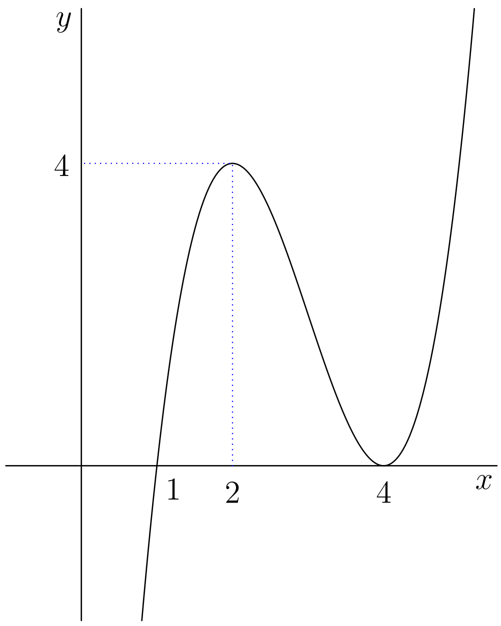
$$x^3 - 9x^2 + 24x - 16 = (x - 1)(x - 4)^2.$$

An alternative method of arriving at the factorization is to note that

$$\begin{aligned}
f(x) &= x^3 - 9x^2 + 24x - 16 \\
&= x^2(x - 1) + x^2 - 9x^2 + 24x - 16 \\
&= x^2(x - 1) - 8x^2 + 24x - 16 \\
&= x^2(x - 1) - 8x(x - 1) - 8x + 24x - 16 \\
&= x^2(x - 1) - 8x(x - 1) + 16x - 16 \\
&= x^2(x - 1) - 8x(x - 1) + 16(x - 1) \\
&= (x - 1)(x^2 - 8x + 16) \\
&= (x - 1)(x - 4)^2.
\end{aligned}$$

Examining the factorization of the polynomial $f(x)$, we note that $f(4) = 0$ and $f(x) \geq 0$ for all real numbers x satisfying $x \geq 1$. It follows that the function 4 attains a local minimum at $x = 4$. (Note that the unbounded interval $[1, +\infty)$ is a neighbourhood of 4 in the set of real numbers: the number 4 cannot be "approached" without "passing through" real numbers satisfying $x \geq 1$.)

The function $f(x)$ does not however attain a *global minimum* at $x = 4$. Indeed if $x < 1$ then $f(x) < 0$, and therefore $f(x) < f(4)$.



Graph of the curve $y = x^3 - 9x^2 + 24x - 16$.

Now various methods (e.g., computer-assisted graphing, numerical investigation, calculus techniques) might suggest that the function attains a local maximum at $x = 2$ with value 4. We can investigate this by dividing the polynomial $f(x)$ by $x - 2$, taking quotient and remainder.

Now

$$f(x) = x^3 - 9x^2 + 24x - 16,$$

and the result of dividing $x^3 - 9x^2 + 24x - 16$ by $x - 2$ may be calculated as follows:—

$$\begin{array}{r|llll} & x^2 & -7x & +10 & \\ \hline x-2 & x^3 & -9x^2 & +24x & -16 \\ & x^3 & -2x^2 & & \\ \hline & & -7x^2 & +24x & \\ & & -7x^2 & +14x & \\ \hline & & & 10x & -16 \\ & & & 10x & -20 \\ \hline & & & & 4 \end{array}$$

This calculation yields the result that

$$x^3 - 9x^2 + 24x - 16 = (x-2)(x^2 - 7x + 10) + 4.$$

Factorizing the quadratic, we find that

$$x^3 - 9x^2 + 24x - 16 = (x-2)^2(x-5) + 4.$$

Alternatively, this result may be obtained as follows:—

$$\begin{aligned} f(x) &= x^3 - 9x^2 + 24x - 16 \\ &= x^2(x-2) + 2x^2 - 9x^2 + 24x - 16 \\ &= x^2(x-2) - 7x^2 + 24x - 16 \\ &= x^2(x-2) - 7x(x-2) - 14x + 24x - 16 \\ &= x^2(x-2) - 7x(x-2) + 10x - 16 \\ &= x^2(x-2) - 7x(x-2) + 10(x-2) + 4 \\ &= (x-2)(x^2 - 7x + 10) + 4 \\ &= (x-2)^2(x-5) + 4. \end{aligned}$$

It follows that $f(x) \leq 4$ whenever $x \leq 5$, and thus the function f does indeed attain a local maximum at 2.