Module MA1S11 (Calculus)
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Section 2: Polynomials

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Contents

2 Polynomials

2.1 Completing the Square in Quadratic Polynomials

A *quadratic polynomial* takes the form

$$ax^2 + bx + c$$

where the *coefficients* a, b and c are numbers (which may be real or complex), and $a \neq 0$.

The qualitative behaviour of a quadratic polynomial and, in particular, the roots of a quadratic polynomial can be determined through a process of "completing the square".

The process of "completing the square", one seeks numbers p and k for which

$$ax^2 + bx + c = a(x - p)^2 + k.$$

Now

$$a(x - p)^2 + k = ax^2 - 2apx + ap^2 + k.$$

On equating coefficients of corresponding powers of x, we arrive at the equations $2ap = -b$ and $ap^2 + k = c$. Solving these equations, we find that

$$p = -\frac{b}{2a} \quad \text{and} \quad k = c - ap^2 = \frac{4ac - b^2}{4a}.$$

A number r is a *root* of the polynomial $ax^2 + bx + c$ if and only if $ar^2 + br + c = 0$. A real number r is thus a root of this polynomial if and only if $a(r - p)^2 = -k$, where

$$p = -\frac{b}{2a} \quad \text{and} \quad k = \frac{4ac - b^2}{4a}.$$

Now a real or complex number w can be determined so that $w^2 = \sqrt{b^2 - 4ac}$. Then

$$-\frac{k}{a} = \frac{b^2 - 4ac}{(2a)^2} = \left(\frac{w}{2a}\right)^2.$$

This number w may then be represented in the form

$$w = \sqrt{b^2 - 4ac}.$$

A root r of the polynomial $ax^2 + bx + c$ must satisfy the equation

$$(r - p)^2 = \left(\frac{w}{2a}\right)^2.$$

It follows that

$$r - p = \pm \frac{w}{2a},$$

and thus

$$r = p \pm \frac{w}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The process of completing the square thus yields the standard formula for the roots of a quadratic polynomial, stated in the following lemma (which follows directly from the immediately preceding remarks).

Lemma 2.1 *Let ax^2+bx+c be a quadratic polynomial, where the coefficients a, b and c are real or complex numbers and $a \neq 0$. Then the roots of the polynomial are given by the formula*

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Lemma 2.2 *Let x^2+bx+c be a quadratic polynomial in which the coefficient of x^2 is equal to one, and let r and s be the roots of the polynomial (with $s = r$ in the case when $b^2 = 4c$). Then $r + s = -b$ and $rs = c$.*

Proof If the roots of the quadratic polynomial are r and s then

$$x^2 + bx + c = (x - r)(x - s) = x^2 - (r + s)x + rs.$$

The result follows. ∎

Remark The result of Lemma 2.2 can be used to check the standard formula for the roots of a quadratic polynomial presented in Lemma 2.1. Indeed a real number x satisfies $ax^2 + bx + c = 0$, where a, b and c are real or complex numbers, with $a \neq 0$, if and only if

$$x^2 + \frac{b}{a}x + \frac{c}{a}.$$

It follows from Lemma 2.2 that real numbers r and s are roots of this quadratic polynomial if and only if

$$r + s = -\frac{b}{a} \quad \text{and} \quad rs = \frac{c}{a}.$$

Let

$$r = \frac{-b + w}{2a} \quad \text{and} \quad s = \frac{-b - w}{2a},$$

where w is some real or complex number (customarily denoted by $\sqrt{b^2 - 4ac}$) that satisfies the equation $w^2 = b^2 - 4ac$. Then

$$r + s = -\frac{b}{a}$$

and

$$\begin{aligned} rs &= \frac{(-b+w)(-b-w)}{4a^2} = \frac{(-b)^2 - w^2}{4a^2} = \frac{b^2 - w^2}{4a^2} \\ &= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}. \end{aligned}$$

It follows that r and s are indeed the roots of the quadratic polynomial $ax^2 + bx + c$.

2.2 Quadratic Polynomials with Real Coefficients

We now restrict our attention to quadratic polynomials $ax^2 + bx + c$ in which the coefficients a, b and c are real numbers and $a \neq 0$. The process of completing the square then yields the equation

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Examining the structure of the formula on the right hand side of the above equation, we can deduce immediately the following result.

Lemma 2.3 *Let ax^2+bx+c be a quadratic polynomial, where the coefficients a, b and c are real numbers and $a > 0$. Then*

$$ax^2 + bx + c \geq \frac{4ac - b^2}{4a}.$$

Moreover

$$ax^2 + bx + c = \frac{4ac - b^2}{4a}$$

if and only if

$$x = -\frac{b}{2a}.$$

To summarize, if the coefficients a, b, c of the quadratic polynomial $ax^2 + bx + c$ are real numbers and $a > 0$, then the quadratic polynomial achieves its minimum value when $x = -b/(2a)$.

Similarly, if the coefficients a, b, c of the quadratic polynomial ax^2+bx+c are real numbers and $a < 0$, then the quadratic polynomial achieves its maximum value when $x = -b/(2a)$.

In both cases determined by the sign of the coefficient a, the minimum value (in the case $a > 0$), or maximum value (in the case $a < 0$), is equal to

$$\frac{4ac - b^2}{4a},$$

Proposition 2.4 *Let a, b and c be real numbers, where $a \neq 0$. Then the sign of the quantity $b^2 - 4ac$ determines the qualitative nature of roots of the quadratic polynomial $ax^2 + bx + c$ according to the following prescription:*

Case when $b^2 > 4ac$: *in this case the polynomial has two distinct real roots;*

Case when $b^2 = 4ac$: *in this case the polynomial has a repeated root at $-b/2a$.*

Case when $b^2 < 4ac$: *in this case the polynomial has two complex roots $p + iq$ and $p - iq$, where*

$$p = -\frac{b}{2a}, \quad q = \frac{4ac - b^2}{2a}, \quad i^2 = -1.$$

2.3 Polynomial Factorization Examples

We discuss examples exemplifying the use of standard methods for solving quadratic equations.

Example We factorize the polynomial

$$x^5 - 13x^3 + 36x.$$

as a product of linear factors of the form $x - r$, where r is some root of the polynomial $p(x)$. Now

$$x^5 - 13x^3 + 36x = x(x^4 - 13x^2 + 36).$$

Moreover

$$x^4 - 13x^2 + 36 = u^2 - 13u + 36,$$

where $u = x^2$. Applying standard methods for finding the roots of quadratic polynomials, we find that

$$u^2 - 13u + 36 = (u - 4)(u - 9).$$

(In this case, the factorization follows directly on noting that 4 and 9 are the unique numbers whose sum is 13 and whose product is 36.) It follows that

$$x^5 - 13x^3 + 36x = x(x^2 - 4)(x^2 - 9).$$

Now

$$x^2 - 4 = (x + 2)(x - 2) \quad \text{and} \quad x^2 - 9 = (x + 3)(x - 3).$$

It follows that

$$x^5 - 13x^3 + 36x = x(x - 1)(x - 2)(x + 3)(x - 3).$$

Example Consider the problem of identifying all non-zero real numbers x that satisfy the equation

$$\frac{1}{x^2} + \frac{2}{x} = 35.$$

There are at least two methods for solving this equation.

To apply the first method, we let $u = 1/x$. Then x satisfies the given equation if and only if the corresponding non-zero real number u satisfies

$$u^2 + 2u - 35 = 0.$$

Now

$$u^2 + 2u - 35 = (u + 7)(u - 5).$$

It follows that the non-zero values of x that solve the equation

$$\frac{1}{x^2} + \frac{2}{x} = 35.$$

are

$$x = -\frac{1}{7} \quad \text{and} \quad x = \frac{1}{5}.$$

To apply the second method, we multiply both sides of the equation

$$\frac{1}{x^2} + \frac{2}{x} = 35$$

by x^2 in order to clear denominators. We find that

$$1 + 2x = x^2\left(1 + \frac{1}{x^2} + \frac{2}{x}\right) = 35x^2.$$

It follows that a non-zero real number x satisfies the equation

$$\frac{1}{x^2} + \frac{2}{x} = 35$$

if and only if it satisfies the quadratic equation

$$35x^2 - 2x - 1 = 0.$$

From the standard quadratic formula, we see that the roots of the polynomial $35x^2 - 2x - 1$ are x_1 and x_2, where

$$x_1 = \frac{2 + \sqrt{4 + 4 \times 35}}{70} \quad \text{and} \quad x_2 = \frac{2 - \sqrt{4 + 4 \times 35}}{70}.$$

Moreover

$$\sqrt{4 + 4 \times 35} = \sqrt{4 \times 36} = \sqrt{4} \times \sqrt{36} = 2 \times 6 = 12.$$

It follows that

$$x_1 = \frac{2 + 12}{70} = \frac{14}{70} = \frac{2}{10} = \frac{1}{5},$$

and

$$x_2 = \frac{2 - 12}{70} = -\frac{10}{70} = -\frac{1}{7}.$$

We have thus found the solutions of the given equation.

Example We now seek to determine all positive real numbers x satisfying the equation

$$x^{\frac{2}{3}} - 5x^{\frac{1}{2}} + 6x^{\frac{1}{3}} = 0.$$

Now

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad \text{and} \quad \frac{2}{3} - \frac{1}{3} = 2 \times \frac{1}{6}.$$

It follows that

$$x^{\frac{2}{3}} - 5x^{\frac{1}{2}} + 6x^{\frac{1}{3}} = x^{\frac{1}{3}}((x^{\frac{1}{6}})^2 - 5x^{\frac{1}{6}} + 6) = x^{\frac{1}{3}}(u^2 - 5u + 6),$$

where $u = x^{\frac{1}{6}}$. Now

$$u^2 - 5u + 6 = (u - 2)(u - 3).$$

It follows that a positive real number x satisfies the equation

$$x^{\frac{2}{3}} - 5x^{\frac{1}{2}} + 6x^{\frac{1}{3}} = 0.$$

if and only if either $x^{\frac{1}{6}} = 2$ or $x^{\frac{1}{6}} = 3$. Therefore the positive real numbers x that satisfy the given equation are 64 and 729.

2.4 Polynomial Division

Example Let $p(x)$ be the polynomial in x defined so that

$$p(x) = x^3 - 8x^2 + 17x - 10.$$

Now $p(1) = 0$. (Indeed the coefficients 1, -8, 17 and -10 add up to zero.) The problem is to find the other roots.

A standard procedure for discovering the other roots is to divide the polynomial $p(x)$ by the polynomial $x-1$ using a calculation scheme modelled on a standard scheme for performing long division in arithmetic. The calculation goes as follows:—

$$\begin{array}{r|rrrr}
 & x^2 & -\,7x & +\,10 & \\ \cline{2-5}
x-1 & x^3 & -\,8x^2 & +\,17x & -\,10 \\
 & x^3 & -\,x^2 & & \\ \cline{2-3}
 & & -7x^2 & +\,17x & \\
 & & -7x^2 & +\,7x & \\ \cline{3-4}
 & & & 10x & -\,10 \\
 & & & 10x & -\,10 \\ \cline{4-5}
 & & & & 0
\end{array}$$

This calculation yields the result that

$$x^3 - 8x^2 + 17x - 10 = (x-1)(x^2 - 7x + 10).$$

Now the polynomial $x^2 - 7x + 10$ can be factored using the standard formula for the roots of a quadratic polynomial. Alternatively, because the leading term is equal to one, it follows from Lemma 2.2 that the sum of the roots of the polynomial $x^2 - 7x + 10$ is equal to 7 and the product of those roots is equal to 10. From this we can deduce that the roots are 2 and 5, and thus

$$x^2 - 7x + 10 = (x-2)(x-5),$$

and thus

$$p(x) = x^3 - 8x^2 + 17x - 10 = (x-1)(x-2)(x-5).$$

We now divide the polynomial $x^3 - 8x^2 + 17x - 10$ by $x - 1$ using standard algebraic notation, to see how the individual steps are justified.

Let

$$p(x) = x^3 - 8x^2 + 17x - 10.$$

First we note that we can obtain a polynomial whose leading term matches the leading term x^3 of $p(x)$ by multiplying the polynomial $x - 1$ by x^2. Now $x^3 = (x - 1)x^2 + x^2$. It follows that

$$\begin{aligned} p(x) &= (x-1)x^2 + x^2 - 8x^2 + 17x - 10 \\ &= (x-1)x^2 - 7x^2 + 17x - 10. \end{aligned}$$

Next we note that we can obtain a polynomial whose leading term is $-7x^2$ by multiplying the polynomial $x - 1$ by $-7x$. Now $-7x^2 = -7(x-1)x - 7x$. It follows that

$$\begin{aligned} p(x) &= x^3 - 8x^2 + 17x - 10 \\ &= (x-1)x^2 - 7x^2 + 17x - 10 \\ &= (x-1)x^2 - 7(x-1)x - 7x + 17x - 10 \\ &= (x-1)(x^2 - 7x) + 10x - 10. \end{aligned}$$

But $10x - 10 = 10(x - 1)$. It follows that

$$p(x) = (x-1)(x^2 - 7x + 10).$$

Moreover $x^2 - 7x + 10 = (x - 2)(x - 5)$. It follows that

$$p(x) = (x-1)(x-2)(x-5).$$

Example We now divide the polynomial

$$ax^3 + bx^2 + cx + d$$

by the polynomial

$$x - r,$$

where the coefficients a, b, c, d and r of these polynomials are numbers (which may be real or complex). The calculation may be set out as a division calculation as follows:

$$
\begin{array}{r|rrrr}
 & & ax^2 & +(ar+b)x & +(ar^2+br+c) \\
\hline
x-r & ax^3 & +bx^2 & +cx & +d \\
 & ax^3 & -arx^2 & & \\
\cline{2-3}
 & & (ar+b)x^2 & +cx & \\
 & & (ar+b)x^2 & -(ar^2+br)x & \\
\cline{3-4}
 & & & (ar^2+br+c)x & +d \\
 & & & (ar^2+br+c)x & -(ar^3+br^2+cr) \\
\cline{4-5}
 & & & & ar^3+br^2+cr+d
\end{array}
$$

This calculation scheme yields the result that

$$
\begin{aligned}
& ax^3 + bx^2 + cx + d \\
& \quad = \quad q(x)(x-r) + ar^3 + br^2 + cr + d,
\end{aligned}
$$

where

$$q(x) = ax^2 + (ar+b)x + (ar^2 + br + c).$$

The following lemma establishes the result more formally, using standard algebraic notation.

Lemma 2.5 *Let $p(x)$ be a polynomial of degree at most 3, given by the formula*

$$p(x) = ax^3 + bx^2 + cx + d,$$

where the coefficients of this polynomial are numbers (which may be real or complex), and let r be a number (which also may be real or complex). Then

$$p(x) = (x-r)q(x) + p(r),$$

where

$$q(x) = ax^2 + (ar+b)x + ar^2 + br + c.$$

Proof

$$
\begin{aligned}
p(x) &= ax^3 + bx^2 + cx + d \\
&= a(x-r)x^2 + arx^2 + bx^2 + cx + d \\
&= a(x-r)x^2 + (ar+b)x^2 + cx + d \\
&= a(x-r)x^2 + (ar+b)(x-r)x + (ar^2+br)x + cx + d
\end{aligned}
$$

$$
\begin{aligned}
&= (x-r)(ax^2 + (ar+b)x) + (ar^2 + br + c)x + d \\
&= (x-r)(ax^2 + (ar+b)x) + (ar^2 + br + c)(x-r) \\
&\qquad + ar^3 + br^2 + cr + d \\
&= (x-r)(ax^2 + (ar+b)x + ar^2 + br + c) + p(r) \\
&= (x-r)q(x) + p(r),
\end{aligned}
$$

as required. ∎

Theorem 2.6 (Remainder Theorem) *Let $p(x)$ be a polynomial of any degree, and let r be a number. Suppose that $q(x)$ is a polynomial and k is a number determined so that*

$$p(x) = q(x)(x-r) + k.$$

Then $k = p(r)$, and thus

$$p(x) = q(x)(x-r) + p(r).$$

Proof The result follows immediately on substituting $x = r$ in the equation $p(x) = q(x)(x-r) + k$. ∎

Theorem 2.7 (Factor Theorem) *Let $p(x)$ be a polynomial of any degree, and let r be a number. Then $x - r$ is a factor of $p(x)$ if and only if $p(r) = 0$.*

Proof If $x - r$ is a factor of $p(x)$ then it follows directly that $p(r) = 0$.

Conversely suppose that $p(r) = 0$. We must prove that $x - r$ is a factor of $p(r)$. Now the Remainder Theorem ensures the existence of a polynomial $q(x)$ such that $p(x) = (x-r)q(x) + p(r)$. But $p(r) = 0$. It follows that $p(x) = (x-r)q(x)$, and thus $x - r$ is a factor of $p(x)$, as required. ∎

The following proposition is useful in limiting the number of cases that need to be considered when given a cubic polynomial with integer coefficients, and it is known that the polynomial already has at least one integer root.

Proposition 2.8 *Let $p(x)$ be a polynomial of degree at most 3, given by the formula*

$$p(x) = ax^3 + bx^2 + cx + d,$$

where the coefficients of this polynomial are integers, and let r be a root of this polynomial that is also an integer. Then r divides d.

Proof The integer r is a root of the polynomial $p(x)$. It follows directly from Lemma 2.5 that

$$p(x) = q(x)(x - r),$$

where

$$q(x) = ax^2 + (ar + b)x + ar^2 + br + c.$$

Equating coefficients, we find that

$$d = -(ar^2 + br + c)r.$$

Now r, a, b, c and d are all integers. It follows that $ar^2 + br + c$ is an integer, and therefore r divides d. The result follows. ∎

Example Consider the polynomial $p(x)$, where

$$4x^3 - 44x^2 + 127x - 105.$$

Now $105 = 3 \times 5 \times 7$, and therefore the divisors of 105 are

$$\pm 1, \ \ \pm 3, \ \ \pm 5, \ \ \pm 7, \ \ \pm 15, \ \ \pm 21, \ \ \pm 35 \ \text{ and } \ \pm 105.$$

Calculating, we find that

$$p(1) = -18, \quad p(-1) = -280,$$

$$\begin{gathered} p(3) = -12, \quad p(-3) = -990, \\ p(5) = -70, \quad p(-5) = -2340, \\ p(7) = 0, \quad p(-7) = -4522, \\ p(15) = 5400, \quad p(-15) = -25410, \\ p(21) = 20202, \quad p(-21) = -59220, \\ p(35) = 121940, \quad p(-35) = -229950, \\ p(105) = 4158630, \quad p(-105) = -5129040. \end{gathered}$$

It follows that 7 is the only root of the polynomial $p(x)$ that is an integer.

Polynomials can always be divided by polynomials of lower degree, taking quotient and remainder. We now give an example of polynomial division that involves dividing a polynomial of degree 4 by a quadratic polynomial.

Example We divide the polynomial $p(x)$ by $x^2 + 2x + 2$, where

$$p(x) = x^4 + 8x^3 + 27x^2 + 39x + 28.$$

The calculation can be undertaken using the following scheme:—

$$\begin{array}{rrrrrrr}
 & & & x^2 & +6x & +13 \\ \cline{2-6}
x^2+2x+2 \big) & x^4 & +8x^3 & +27x^2 & +39x & +28 \\
 & x^4 & +2x^3 & +2x^2 & & \\ \cline{2-4}
 & & 6x^3 & +25x^2 & +39x & \\
 & & 6x^3 & +12x^2 & +12x & \\ \cline{3-5}
 & & & 13x^2 & +27x & +28 \\
 & & & 13x^2 & +26x & +26 \\ \cline{4-6}
 & & & & x & +2
\end{array}$$

This calculation scheme yields the result that

$$x^4 + 8x^3 + 27x^2 + 39x + 28 = (x^2 + 2x + 2)(x^2 + 6x + 13) + x + 2.$$

We may establish this result using standard algebraic notation as follows:—

$$\begin{aligned}
p(x) &= x^4 + 8x^3 + 27x^2 + 39x + 28 \\
&= (x^2 + 2x + 2)x^2 - 2x^3 - 2x^2 + 8x^3 + 27x^2 + 39x + 28 \\
&= (x^2 + 2x + 2)x^2 + 6x^3 + 25x^2 + 39x + 28 \\
&= (x^2 + 2x + 2)x^2 + 6(x^2 + 2x + 2)x - 12x^2 - 12x \\
&\qquad + 25x^2 + 39x + 28 \\
&= (x^2 + 2x + 2)(x^2 + 6x) + 13x^2 + 27x + 28 \\
&= (x^2 + 2x + 2)(x^2 + 6x) + 13(x^2 + 2x + 2) - 26x - 26 \\
&\qquad + 27x + 28 \\
&= (x^2 + 2x + 2)(x^2 + 6x + 13) + x + 2.
\end{aligned}$$

Thus

$$p(x) = (x^2 + 2x + 2)(x^2 + 6x + 13) + x + 2.$$

The structure of calculation in the standard scheme can also be clarified by adding redundant terms (coloured red) as follows:—

$$
\begin{array}{rrrrrr}
 & & & x^2 & +\,6x & +\,13 \\
\cline{2-6}
x^2+2x+2 \big) & x^4 & +\,8x^3 & +\,27x^2 & +\,39x & +\,28 \\
 & x^4 & +\,2x^3 & +\,2x^2 & \color{red}{+0x} & \color{red}{+0} \\
\cline{2-6}
 & & 6x^3 & +\,25x^2 & +\,39x & \color{red}{+28} \\
 & & 6x^3 & +\,12x^2 & +\,12x & \color{red}{+0} \\
\cline{3-6}
 & & & 13x^2 & +\,27x & +\,28 \\
 & & & 13x^2 & +\,26x & +\,26 \\
\cline{4-6}
 & & & & x & +\,2
\end{array}
$$