Investigation of the Regular Pentagon using Complex Numbers

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1.1 Basic Properties of the Complex Numbers

Sir William Rowan Hamilton, in a presentation delivered to the British Association for the Advancement of Science, at its meeting at Trinity College Dublin in 1835, expounded his approach to the understanding of the basic principles underlying the system of complex numbers. He argued that "complex numbers" should be identified with *algebraic couples* of the form (x, y), where x and y are real numbers, and where addition and multiplication of algebraic couples are *defined* by the identities:

$$(x, y) + (u, v) = (x + u, y + v), \quad (x, y) \times (u, v) = (xu - yv, xv + yu).$$

The resulting algebraic system has the properties that result from these definitions, when the normal "rules of algebra", such as the Commutative and Distributive Laws, are applied. Consequently

$$(x, y) - (u, v) = (x - u, y - v), \quad (x, y) \div (u, v) = \left(\frac{xu + yv}{u^2 + v^2}, \frac{yu - xv}{u^2 + v^2} \right),$$

where division of algebraic couples is defined in cases, and only in cases, where the real and imaginary parts, u and v of the denominator are not both equal to zero.

In this system of "algebraic couples" we identify each real number x with the algebraic couple $(x, 0)$, and we set $i = (0, 1)$. The rules for multiplying algebraic couples then ensure that

$$i^2 = (0, 1)^2 = (0, 1) \times (0, 1) = (-1, 0) = -1.$$

Moreover

$$x + iy = (x, 0) + ((0, 1) \times (y, 0)) = (x, 0) + (0, y) = (x, y).$$

We thus obtain the standard representation of complex numbers, in which complex numbers are expressed in the from $x + iy$, where x and y are real numbers and $i^2 = -1$.

The *complex conjugate* of a complex number represented as an algebraic couple (x, y) is that represented as $(x, -y)$. The complex conjugate of a complex number z is denoted by $\overline{z}$.

Let z and w be complex numbers represented in Hamilton's scheme by algebraic couples (x, y) and (u, v) respectively. Then

$$\overline{z} + \overline{w} = (x, -y) + (u, -v) = (x + u, -(y + v)) = \overline{z + w}$$

and

$$\overline{z}\,\overline{w} = (x, -y) \times (u, -v) = (xu - yv, -(xv + yu)) = \overline{zw}.$$

Also

$$z\overline{z} = (x, y) \times (x, -y) = (x^2 + y^2, 0) = |z|^2,$$

where $|x + iy| = \sqrt{x^2 + y^2}$ for all real numbers x and y. It follows, on substituting $-y$ for y, that $\overline{z}z = |z|^2$.

Now let complex numbers z and w satisfy $|z| = 1$ and $|w| = 1$, and let $z = x + iy$ and $w = u + iv$, where x, y, u and v are real numbers. Then $x^2 + y^2 = 1$ and $u^2 + v^2 = 1$, and consequently real numbers θ and ψ can be found so that

$$z = \cos\theta + i\,\sin\theta, \quad w = \cos\psi + i\,\sin\psi.$$

Then

$$\begin{aligned} zw &= \cos\theta\,\cos\psi - \sin\theta\,\sin\psi + (\sin\theta\,\cos\psi + \cos\theta\,\sin\psi) \\ &= \cos(\theta + \psi) + i\sin(\theta + \psi). \end{aligned}$$

Applying this identity with $\psi = (m - 1)\theta$ for $m = 1, 2, 3, \ldots$, we find that

$$(\cos\theta + i\,\sin\theta)^m = \cos m\theta + i\sin m\theta$$

for all positive integers m. Moreover

$$(\cos\theta + i\,\sin\theta)^{-1} = \cos\theta - i\sin\theta.$$

Consequently

$$\begin{aligned} (\cos\theta + i\,\sin\theta)^{-m} &= (\cos\theta - i\sin\theta)^m = \cos m\theta - i\sin m\theta \\ &= \cos(-m\theta) + i\sin(-m\theta) \end{aligned}$$

for all positive integers m. Combining the results obtained above, and using the identities $\sin 0 = 0$ and $\cos 0 = 1$, we find that

$$(\cos\theta + i\,\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers n, be those integers positive, negative or zero. This result is *De Moivre's Theorem.*

1.2 The Fifth Roots of Unity in the Complex Plane

The complex numbers z that satisfy the equation $z^5 = 1$ constitute the vertices of a regular pentagon inscribed in the unit circle in the complex plane. Indeed if $z^5 = 1$ then $|z|^5 = |z^5| = 1$, and therefore $|z| = 1$. It follows that the solutions of the equation $z^5 = 1$ in the complex plane all lie on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane. It follows that if the complex number z satisfies the equation $z^5 = 1$ and if $z = x + iy$, where x and y are real numbers and $i^2 = -1$, then $x^2+y^2 = 1$, and therefore $x = \cos\theta$ and $y = \sin\theta$ for some real number θ, and consequently

$$z = \cos\theta + i\sin\theta.$$

It then follows from De Moivre's Theorem that

$$1 = z^5 = \cos 5\theta + i\sin 5\theta,$$

and therefore $5\theta = 2\pi n$ for some integer n. Consequently the complex numbers satisfying the equation $z^5 = 1$ are the complex numbers

$$1, \omega, \omega^2, \omega^3 \text{ and } \omega^4,$$

where

$$\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}.$$

The angle here represented in radian measure as $\frac{2}{5}\pi$ is the angle of 72°. It is the angle between lines joining two succesive vertices of a regular pentagon to the centre of that pentagon.

Next we note that if the complex number z satisfies the equation $z^5 = 1$, and if $z \neq 1$, then

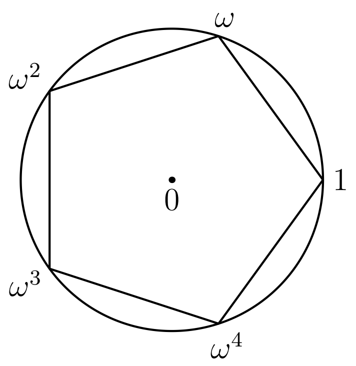
$$\begin{aligned} &(1-z)(1+z+z^2+z^3+z^4) \\ &\qquad = (1-z)+(z-z^2)+(z^2-z^3)+(z^3-z^4)+(z^4-z^5) \\ &\qquad = 1-z^5 = 0, \end{aligned}$$

and therefore

$$1 + z + z^2 + z^3 + z^4 = 0.$$

Consequently the complex numbers ω, ω^2, ω^3 and ω^4 are the four roots of the polynomial

$$1 + z + z^2 + z^3 + z^4.$$



1.3 Some Calculations involving Fifth Roots of Unity

First we note that

$$\omega \,.\, \omega^4 = \omega^5 = 1 = \omega \,.\, \overline{\omega}.$$

Dividing by ω, we find that $\omega^4 = \overline{\omega}$. Thus if $\omega = u + iv$, where $u = \cos(2\pi/5)$ and $v = \sin(2\pi/5)$, then $\omega^4 = u - iv$. Similarly

$$\omega^2 \,.\, \omega^3 = \omega^5 = 1 = \omega^2 \,.\, \overline{\omega^2}.$$

Dividing by ω^2, we find that $\omega^3 = \overline{\omega^2}$. Thus if $\omega^2 = -s + it$, where

$$s = -\cos\left(\frac{4\pi}{5}\right) = \cos\left(\pi - \frac{4\pi}{5}\right) = \cos\left(\frac{\pi}{5}\right)$$

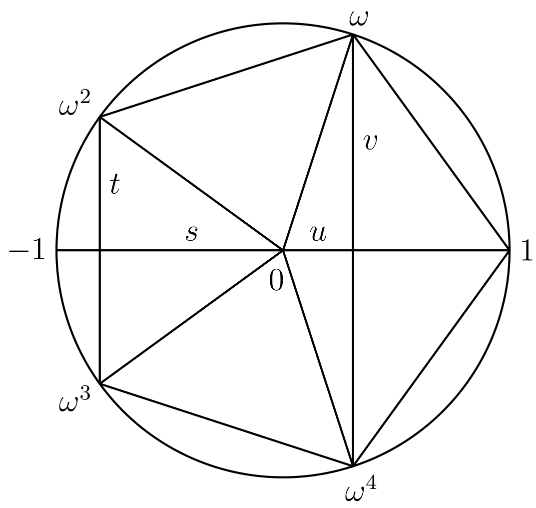
and

$$t = \sin\left(\frac{4\pi}{5}\right) = \sin\left(\pi - \frac{4\pi}{5}\right) = \sin\left(\frac{\pi}{5}\right),$$

then $\omega^3 = -s - it$.

The results obtained above may therefore be summarized in the statement that

$$\omega = u + iv, \quad \omega^2 = -s + it, \quad \omega^3 = -s + it \quad \text{and} \quad \omega^4 = u - iv,$$



where

$$u = \cos\frac{2\pi}{5} = \cos 72^\circ, \quad v = \sin\frac{2\pi}{5} = \sin 72^\circ,$$
$$s = \cos\frac{\pi}{5} = \cos 36^\circ, \quad t = \sin\frac{\pi}{5} = \sin 36^\circ.$$

We now examine the squares of the complex numbers $\omega \pm \omega^4$ and $\omega^2 \pm \omega^3$. We find that

$$\begin{aligned}(\omega + \omega^4)^2 &= \omega^2 + 2\omega^5 + \omega^8 = \omega^2 + 2 + \omega^3,\\ (\omega - \omega^4)^2 &= \omega^2 - 2\omega^5 + \omega^8 = \omega^2 - 2 + \omega^3,\\ (\omega^2 + \omega^3)^2 &= \omega^4 + 2\omega^5 + \omega^6 = \omega^4 + 2 + \omega,\\ (\omega^2 - \omega^3)^2 &= \omega^4 - 2\omega^5 + \omega^6 = \omega^4 - 2 + \omega.\end{aligned}$$

Consequently

$$(\omega + \omega^4)^2 + \omega + \omega^4 - 1 = 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$$

and

$$(\omega^2 + \omega^3)^2 + \omega^2 + \omega^3 - 1 = 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0,$$

and therefore

$$4u^2 + 2u - 1 = 0 \quad \text{and} \quad 4s^2 - 2s - 1 = 0.$$

Solving these quadratic equations by the usual methods, and taking note of the inequalities $u > 0$ and $s > 0$, we find that

$$u = \frac{-1 + \sqrt{5}}{4} \quad \text{and} \quad s = \frac{1 + \sqrt{5}}{4}.$$

Next let

$$\varphi = \frac{\omega - \omega^4}{\omega^2 - \omega^3}.$$

Now $\omega - \omega^4 = 2iv$ and $\omega^2 - \omega^3 = 2it$, where v and t have the values specified above. It follows that φ is a positive real number, and $\varphi = v/t$. Moreover $\omega = \omega^6$, because $\omega^5 = 1$, and therefore

$$\varphi = \frac{\omega^6 - \omega^4}{\omega^2 - \omega^3} = \frac{(\omega^3 - \omega^2)(\omega^3 + \omega^2)}{\omega^2 - \omega^3} = -(\omega^2 + \omega^3) = 2s = \frac{1 + \sqrt{5}}{2}.$$

Now we have noted above that $4s^2 - 2s - 1 = 0$. Consequently $\varphi^2 - \varphi - 1 = 0$. Thus $\varphi^2 = \varphi + 1$, and therefore

$$\frac{\varphi + 1}{\varphi} = \varphi.$$

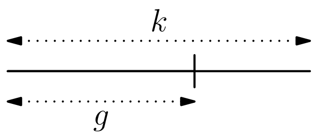
1.4 The Golden Section

Consider a given straight line segment that is divided by a point in the interior of the segment into a greater or lesser segment in such a way as to ensure that the ratio of the length of the whole to that of the greater segment is equal to the ratio of length of the greater segment of the lesser segment. Suppose that the given straight line segment has length k and that the greater segment has length g. Then

$$\frac{k}{g} = \frac{g}{k - g}$$

and therefore

$$\left(\frac{k}{g} - 1\right)\frac{k}{g} = 1.$$



Rearranging, we find that

$$\left(\frac{k}{g}\right)^2 = \frac{k}{g} + 1.$$

Thus the ratio k/g is a root of the same quadratic polynomial as the number φ. That polynomial has a unique positive root. It follows that

$$\frac{k}{g} = \varphi.$$

In situations where the given straight line segment is divided into a greater and lesser segment in the manner described above, the ancient Greeks would say that the straight line segment has been cut *in extreme and mean ratio*. (See Euclid's *Elements of Geometry*, Book VI, Definition 3.) From the early nineteenth century onwards, it has become customary to refer to this particular ratio represented by the real number φ as the *Golden Section* ((The introduction of this name, *goldener Schnitt*, in the German language, is attributed to the mathematician Martin Ohm.) But, although the name *Golden Section* only dates from the early nineteenth century, the properties of this ratio had been much studied in preceding centuries, and indeed plays an important role in several books of Euclid's *Elements of Geometry*.

1.5 Cartesian Coordinates of the Vertices of the Regular Pentagon

We resume our investigation of the geometry of the regular pentagon, determining expressions for the Cartesian coordinates of the vertices of the regular pentagon in inscribed in the unit circle, expressing those coordinates in terms of the number φ that represents the ratio that is named the Golden Section.

Now $\omega^2 = -s + it$, where $s = \frac{1}{2}\varphi$, where $\varphi^2 = \varphi + 1$. Moreover $s^2 + t^2 = 1$. It follows that

$$t^2 = \tfrac{1}{4}(4 - \varphi^2) = \tfrac{1}{4}(3 - \varphi) = \tfrac{1}{8}(5 - \sqrt{5}).$$

Now we also showed above that $\varphi = v/t$. Consequently

$$\begin{aligned} v^2 &= \varphi^2 t^2 = (\varphi + 1)t^2 = \tfrac{1}{4}(\varphi + 1)(3 - \varphi) = \tfrac{1}{4}(3 + 2\varphi - \varphi^2) = \tfrac{1}{4}(2 + \varphi) \\ &= \tfrac{1}{8}(5 + \sqrt{5}). \end{aligned}$$

Also

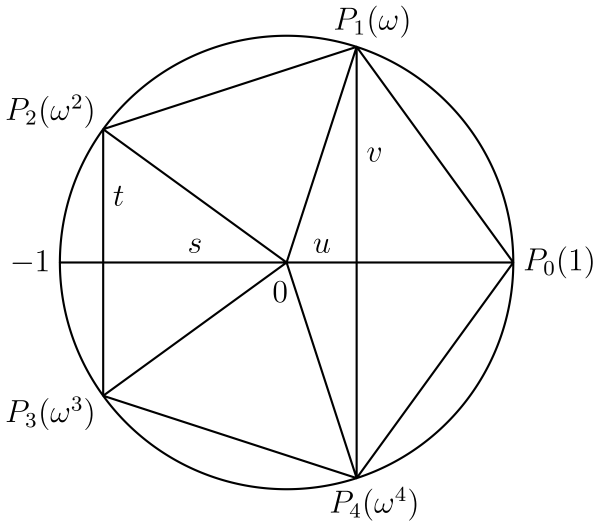
$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0,$$

where $\omega + \omega^4 = 2u$ and $\omega^2 + \omega^3 = -2s$. It follows that $1 + 2u - 2s = 0$, and therefore

$$u = s - \frac{1}{2} = \tfrac{1}{2}(\varphi - 1).$$

If we use to correspondence between complex numbers and points of the plane $\mathbb{R}^2$, using the standard Argand diagram, we find that, given a regular pentagon inscribed in the unit circle $x^2 + y^2 = 1$ in the plane $\mathbb{R}^2$, where one vertex of that pentagon is located at the point P_0, where $P_0 = (1, 0)$, then the other vertices are located at points P_1, P_2, P_3 and P_4, where

$$\begin{aligned} P_1 &= \left(\tfrac{1}{2}(\varphi - 1), \tfrac{1}{2}\sqrt{2+\varphi}\right) = \left(\frac{\sqrt{5}-1}{4}, \sqrt{\frac{5+\sqrt{5}}{8}}\right), \\ P_2 &= \left(\tfrac{1}{2}\varphi, \tfrac{1}{2}\sqrt{3-\varphi}\right) = \left(\frac{\sqrt{5}+1}{4}, \sqrt{\frac{5-\sqrt{5}}{8}}\right), \\ P_3 &= \left(\tfrac{1}{2}\varphi, -\tfrac{1}{2}\sqrt{3-\varphi}\right) = \left(\frac{\sqrt{5}+1}{4}, -\sqrt{\frac{5-\sqrt{5}}{8}}\right), \\ P_4 &= \left(\tfrac{1}{2}(\varphi - 1), -\tfrac{1}{2}\sqrt{2+\varphi}\right) = \left(\frac{\sqrt{5}-1}{4}, -\sqrt{\frac{5+\sqrt{5}}{8}}\right). \end{aligned}$$



Next we note that the length p of a side of this regular pentagon is the distance between the vertices P_2 and P_3, and consequently

$$p = 2t = \sqrt{\frac{5-\sqrt{5}}{2}}.$$

Next we determine the length of a side d of a regular decagon inscribed in the unit circle. Note that P_2 and $(-1, 0)$ are two successive vertices of this regular decagon, where $P_2 = (-s, t)$. Applying Pythagoras' Theorem, and using the identity $s^2 + t^2 = 1$, we find that

$$d^2 = (1-s)^2 + t^2 = 1 - 2s + s^2 + t^2 = 2 - 2s = 2 - \varphi.$$

Consequently

$$d^2 + 1 = 3 - \varphi = 4t^2 = p^2.$$

Now the length h of the side of a regular hexagon inscribed in the unit circle satisfies $h = 1$. Consequently $d^2 + h^2 = p^2$. The corresponding equality holds in circles of any size. This establishes the following proposition:

> *If an equilateral pentagon be inscribed in a circle, the square on the side of the pentagon is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the same circle.*

This proposition is Proposition 10 in Book XIII of Euclid's *Elements of Geometry*.

There are other propositions in Books II, IV and XIII of Euclid's elements that state results that are geometrical analogues of many of the identities established algebraically above.

1.6 Investigations concerning the Pentagon in Euclid's *Elements of Geometry*

In particular Proposition 11 in Book IV of Euclid's *Elements* describes and justifies the construction of a regular pentagon inscribed in a circle that can be achieved using straightedge and compass. The preceding proposition describes and justifies the construction of an isosceles triangle in which the angles at the base are double the angle at the vertex. The angle at the vertex of this triangle is 36° (i.e., the angle between successive vertices of a regular decagon inscribed in a circle, when viewed from the centre of that circle. The ratio of the length of the sides of this particular isosceles triangle is the Golden Section ratio φ discussed above.

Proposition 10 in Book XIII of Euclid's *Elements* is used in the proof the result of Proposition 16 in the same book. Suppose we are given 20 equilateral triangles of the same size. Five of those triangles can be attached together along edges to form a cap at whose vertex all the triangles meet. Five more of those triangles can be attached together to form a corresponding base of the same shape. The remaining ten equilateral triangles can be attached together to form a belt. The cap and base can then be attached to the belt so constructed to form an isosahedron. Now Proposition 16 in Book XIII of Euclid's *Elements* establishes that the vertices of this isosahedron lie on a sphere.

Sir Thomas L. Heath's *Historical Note* that forms a preface to his translation of Book XIII of Euclid's *Elements* commences as follows:

I have already given, in the note to IV. 10, the evidence upon which the construction of the five regular solids is attributed to the Pythagoreans. Some of them, the cube, the tetrahedron (which is nothing but a pyramid), and the octahedron (which is only a double pyramid with a square base), cannot but have been known to the Egyptians. And it appears that dodecahedra have been found, of bronze or other material, which may belong to periods earlier than Pythagoras' time by some centuries (for references see Cantor's *Geschichte der Mathematik*, I_3, pp. 175–6).

It is true that the author of the scholium No. 1 to Eucl. XIII. says that the Book is about "the five so-called Platonic figures, which however do not belong to Plato, three of the aforesaid five figures being due to the Pythagoreans, namely the cube, the pyramid and the dodecahedron, while the octahedron and the icosahedron are due to Theaetetus." This statement (taken probably from Geminus) may perhaps rest on the fact that Theaetetus was the first to write at any length about the two last-mentioned solids. We are told indeed by Suidas (s. v. Θεαίτητος) that Theaetetus "first wrote on the 'five solids' as they are called." This no doubt means that Theaetetus was the first to write a complete and systematic treatise on all the regular solids; it does not exclude the possibility that Hippasus or others had already written on the dodecahedron. The fact that Theaetetus wrote upon the regular solids agrees very well with the evidence which we possess of his contributions to the theory of irrationals, the connexion between which and the regular solids is seen in Euclid's Book XIII.

Theaetetus flourished about 380 B.C, and his work on the regular solids was soon followed by another, that of Aristaeus, an elder contemporary of Euclid, who also wrote an important book on *Solid Loci*, i.e. on conics treated as loci. This Aristaeus (known as "the elder") wrote in the period about 320 B.C. We hear of his *Comparison of the five regular solids* from Hypsicles of the *Elements* as Book XIV. Hypsicles gives in this Book some six propositions supplementing Eucl. XIII.; and he introduces second of the propositions (Heiberg's Euclid, Vol. V. p. 6) as follows:

> "*The same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron when both are inscribed in the same sphere.* This is proved by

Aristaeus in the book entitled *Comparison of the five figures*."

Hypsicles proceeds (pp. 7 sqq.) to give a proof of this theorem. Allman pointed out (*Greek Geometry from Thales to Euclid*, 1889, pp. 201–2) that this proof depends on eight theorems, six of which appear in Euclid's Book XIII. (in Propositions 8, 10, 12, 15, 16 with Por. 17); two other propositions not mentioned by Allman are also used, namely XIII. 4 and 9. This seems, as Allman says, to confirm the inference of Bretschneider (p. 171) that, as Aristaeus' work was the newest and latest in which, before Euclid's time, this subject was treated, we have in Eucl. XIII. at least a partial recapitulation of the contents of the treatise of Aristaeus.

After Euclid, Apollonius wrote on the comparison of the dodecahedron and the icosahedron inscribed in one and the same sphere. This we also learn from Hypsicles, who says in the next words following those about Aristaeus above quoted: "But it is proved by Apollonius in the second edition of his *Comparison of the dodecahedron with the icosahedron* that, as the surface of the dodecahedron is to the surface of the icosahedron [inscribed in the same sphere], so is the dodecahedron itself [i.e., its volume] to the isosahedron, because the perpendicular is the same from the centre of the sphere to the pentagon of the dodecahedron and to the triangle of the icosahedron.

Note, in the above historical note, the detailed metrical information concerning the geometry of the pentagon, the icosahedron and the dodecahedron that, on the evidence of the quoted texts, is to be attributed to the ancient Greeks.