Study Note—Euclid's *Elements*, Book I, Proposition 8

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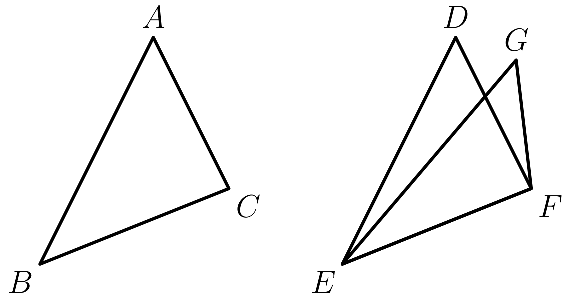
Proposition 8 of Book I of Euclid's *Elements of Geometry* encapsulates the SSS Congruence Rule. The proposition ensures that if the sides BC, CA and AB of a triangle ABC are respectively equal to the sides EF, FD and DE of a triangle DEF, then the angles of the first triangle at vertices A, B and C are respectively equal to the angles of the second triangle at D, E and F.

Euclid supposes that the first triangle could in principle be moved, without changing the lengths of the sides, or the angles at the vertices, so as move the vertices B and C of the first triangle to the locations of the vertices E and F of the second triangle, whilst ensuring that the vertex C of the first triangle is moved to a location on the same side of the side EF of the second triangle as the vertex D of that second triangle.



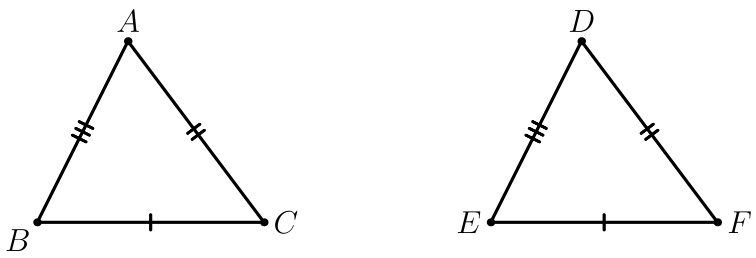
On applying the preceding Proposition 7, it follows that, when the first triangle ABC is moved in this fashion, the vertex A is moved to the position where the vertex D of the second triangle is located. Consequently the first triangle can be moved, without altering the lengths of its sides and the angles at its vertices, so as to coincide exactly with the second triangle. Consequently the angles of the first triangle must be equal to the corresponding angles of the second.

The Alternative Proof of Proposition 8 attributed to Philo of Byzantium.

Proclus, in his commentary on Proposition 8 of the first book of Euclid's *Elements of Geometry* supplies an alternative proof of Proposition 8, which he attributes to Philo of Byzantium (who lived in the third century before the common era).

Philo's proof runs as follows.

Let ABC and DEF be triangles for which the sides BC, CA and AB are equal in length to the sides EF, FD and DE respectively of the second triangle, as indicated in the following figure:



We must show that the angles of the first triangle at A, B and C are respectively equal to the angles of the second triangle at D, E and F.

The proof strategy attributed to Philo of Byzantium requires a triangle to be placed on the side EF of the second triangle, and on the opposite side to the vertex D of the second triangle, so that the triangle GEF so placed is equal to in all respects to the triangle ABC. It should thus be possible to move the triangle ABC, without altering the lengths of its sides, or the angles of its vertices, so that the resulting triangle has vertices at G, E and F, where the sides EF, FG and GE of the new triangle are respectively equal in length to the respective sides BC, CA and AB of the first given triangle, and the angles of the new triangle at G, E and F are respectively equal to the angles of the first given triangle at A, B and C. The points D and G are then joined by a finite straight line.

Five cases arise, depending on the location of the point where the finite straight line DG intersects the infinite straight line that passes through the points E and F:

(i) the first case arises when the straight line DG passes though the point F;

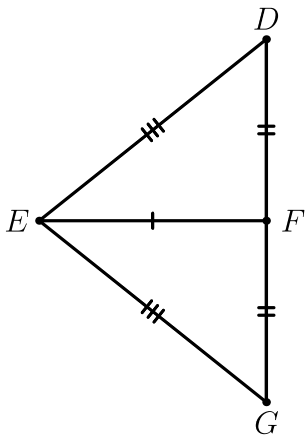
(ii) the second case arises when the straight line DG passes though a point on EF lying between E and G;

(iii) the third case arises when the straight line DG intersects the infinite straight line that passes through E and F at a point on the opposite side of the point F to the point E;

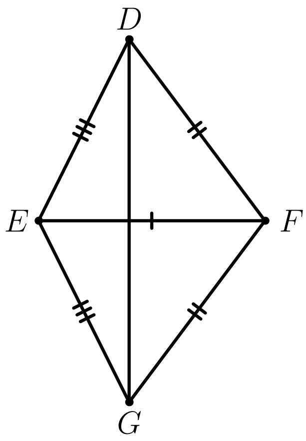
(iv) the fourth case arises when the straight line DG passes through the point E, and the proof in this case is exactly analogous to that in the first case;

(v) the fifth case arises when the straight line DG intersects the infinite straight line that passes through E and F at a point on the opposite side of the point E to the point F, and the proof in this case is exactly analogous to that in the third case.

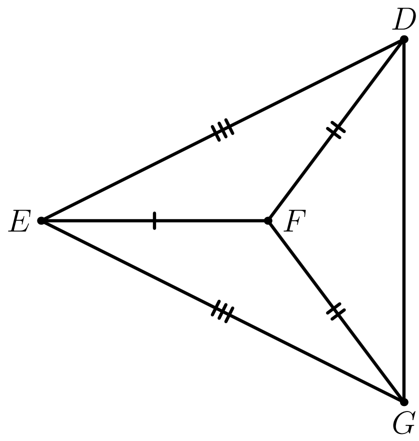
The first three cases are discussed by Proclus in his commentary on Proposition 8 of Book I of Euclid's *Elements of Geometry*. The results in the last two cases may be deduced immediately from those in the first and third cases on interchanging the labels on the vertices so that vertices B, C, E and F are relabelled C, B, F and E respectively.



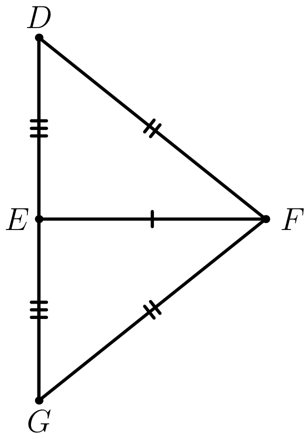
(i)



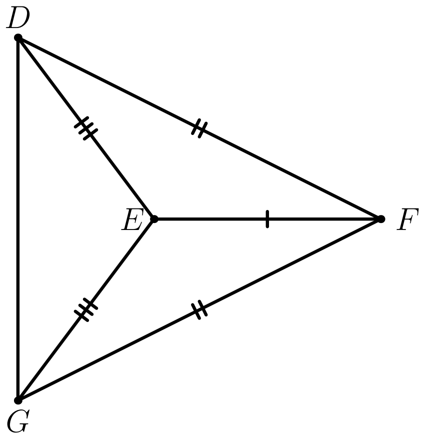
(ii)



(iii)

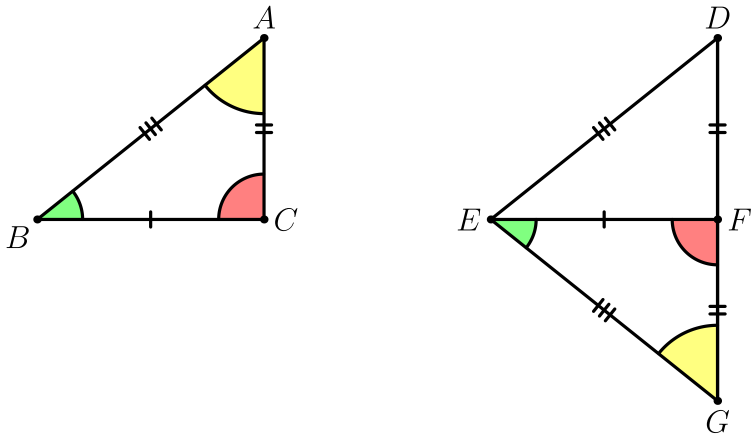


(iv)



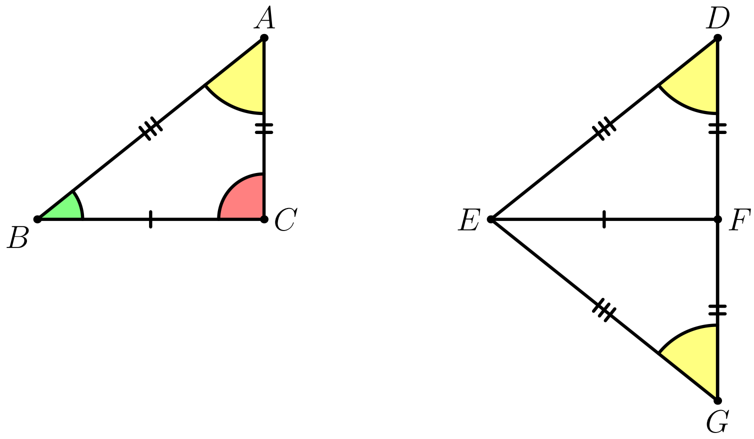
(v)

We first prove the result in case (i). The configuration in this case is as follows.

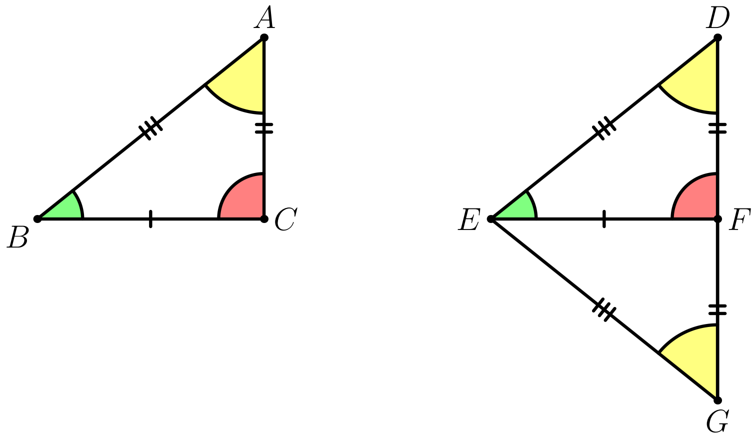


Thus sides BC and EF are equal to one another, sides AC, DF and GF are equal to one another, sides AB, DE and GE are equal to one another, angles CAB and FGE are equal to one another, angles ABC and GEF are equal to one another, and angles BCA and EFG are equal to one another.

Now ADG is an isosceles triangle with base DG and equal sides AD and AG. It follows from the Isosceles Triangle Theorem (Euclid, *Elements*, I.5, also known as the *Pons Asinorum*), that the angles EDG and EGD (coloured in yellow) at the base of this isosceles triangle are equal to one another, and consequently the angle EDF is equal to the angle EGF (given that, in this case, the point F lies on the finite straight line joining the points D and G. But the angle BAC is also equal to the angle EGF, and angles that are equal to the same angle are equal to one another (applying the first of Euclid's common notions). Consequently the angles BAC and EDF are equal to one another.



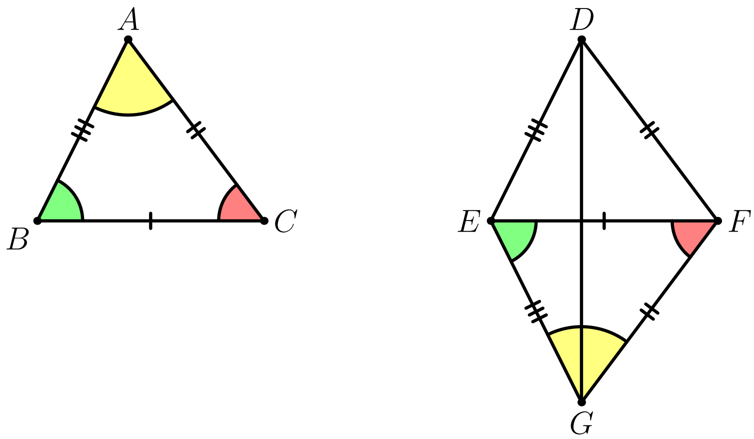
We now apply the SAS Congruence Rule (Euclid, *Elements*, I.4) to the triangles BAC and EDF. The sides BA and AC are equal to the sides ED and DF respectively. Moreover we have shown that the angle BAC enclosed by the first pair of sides is equal to the angle EDF enclosed by the second pair of sides.



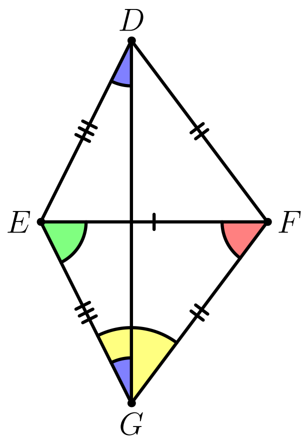
Applying the SAS Congruence Rule (Euclid, *Elements*, I.4), we conclude that the angles of the triangle BAC are equal to the corresponding angles of the triangle EDF and thus the angles ABC and DEF are equal to one another, and also the angles BCA and EFD are equal to one another. This completes the proof of Proposition 8 in case (i) in which the straight line DG passes through the point F.

We next prove the result in case (ii). In this case the straight lines DG and EF intersect at a point that lies between E and F. Thus the configuration in this case is as follows.

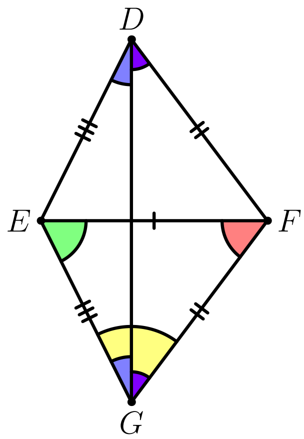
This configuration is constructed so that the sides and angles of the triangle ABC are equal to the corresponding sides and angles of the triangle GEF, so that the sides BC, CA and AB of the triangle ABC are respectively equal to the sides EF, FG and GA of the triangle GEF, and the angles of the first triangle at A, B and C are respectively equal to the angles of the second triangle at G, E and F.



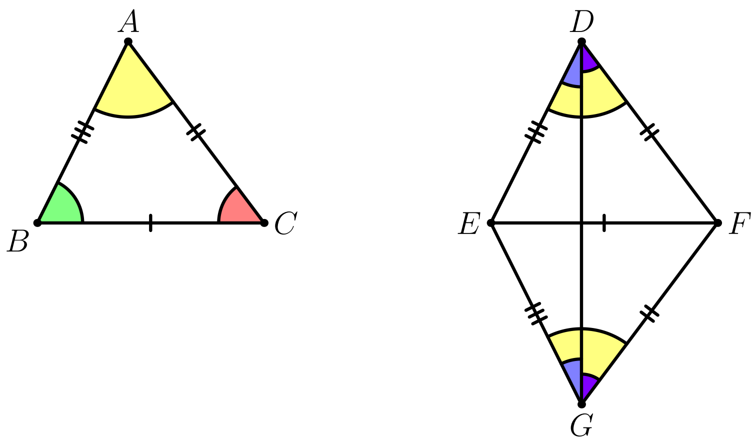
The triangle EDG is then an isosceles triangle with base DG and equal sides ED and EG. The angles EDG and EGD of this triangle at vertices D and G are therefore equal to one another by the Isosceles Triangle Theorem (Euclid, *Elements*, Proposition 5).



Similarly the triangle FDG is an isosceles triangle with base DG. Therefore the angles FDG and FGD are equal to one another.

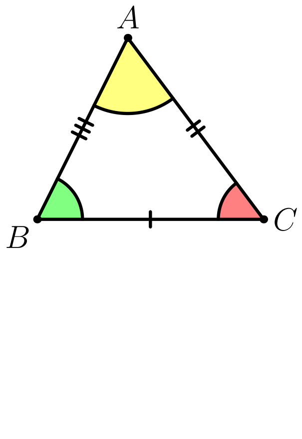


Now the angle EDF is the sum of the angles EDG and FDG, and the angle EGF is the sum of the angles EGD and FGD. We have shown that the angles EDG and FDG are respectively equal to the angles EGD and FGD. Also, by the second of Euclid's common notions, when equals are added to equals, the sums are themselves equal. It therefore follows that the angles EDF and EGF (coloured in yellow) are equal to one another.



We now apply the SAS Congruence Rule (Euclid, *Elements*, I.4) to the triangles BAC and EDF. The sides BA and AC are equal to the sides ED and DF respectively. Moreover we have shown that the angle BAC enclosed by the first pair of sides is equal to the angle EDF enclosed by the second pair of sides.

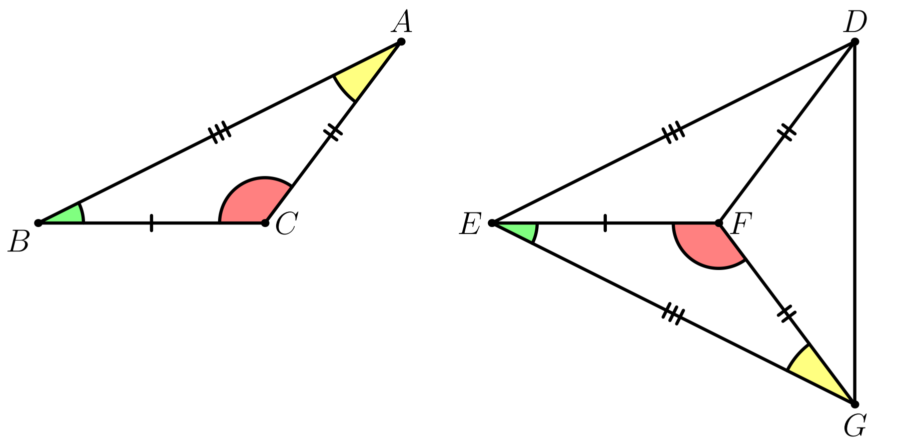
Applying the SAS Congruence Rule (Euclid, *Elements*, I.4), we conclude that the angles of the triangle BAC are equal to the corresponding angles



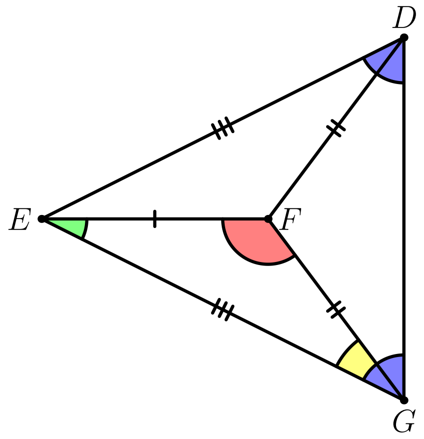
of the triangle EDF and thus the angles ABC and DEF are equal to one another, and also the angles BCA and EFD are equal to one another. This completes the proof of Proposition 8 in case (ii) in which the straight lines DG and EF intersect at a point lying between the points E and F.

Finally we prove the result in case (iii). In this case the straight line DG intersects the infinite straight line that passes through the points E and F as some intersection point that lies on the opposite side of the point F to the point E. Thus the configuration in this case is as follows.

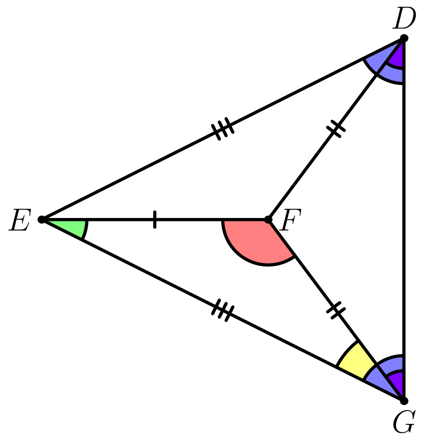
This configuration is constructed so that the sides and angles of the triangle ABC are equal to the corresponding sides and angles of the triangle GEF, so that the sides BC, CA and AB of the triangle ABC are respectively equal to the sides EF, FG and GA of the triangle GEF, and the angles of the first triangle at A, B and C are respectively equal to the angles of the second triangle at G, E and F.



The triangle EDG is then an isosceles triangle with base DG and equal sides ED and EG. The angles EDG and EGD of this triangle at vertices D and G are therefore equal to one another by the Isosceles Triangle Theorem (Euclid, *Elements*, Proposition 5).

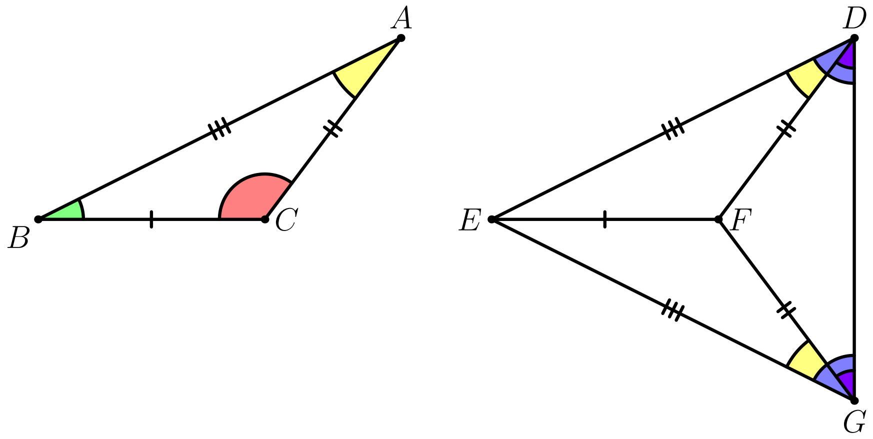


Similarly the triangle FDG is an isosceles triangle with base DG. Therefore the angles FDG and FGD are equal to one another.

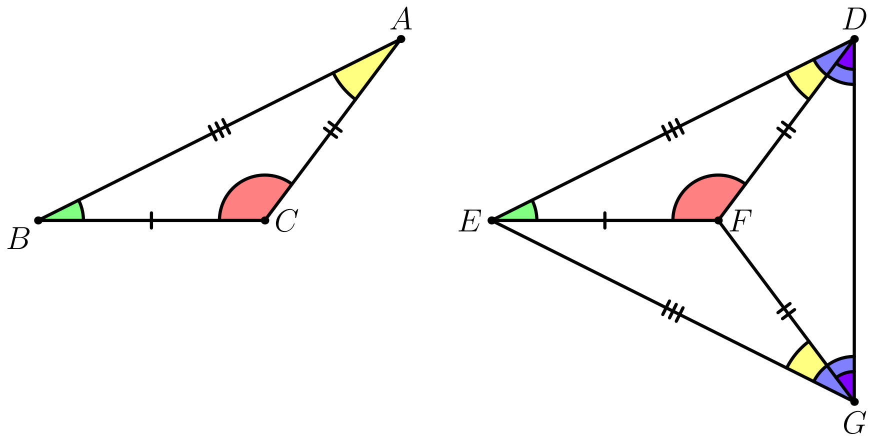


Now the angle EDF is the difference of the angles EDG and FDG, and the angle EGF is the difference of the angles EGD and FGD. We have shown that the angles EDG and FDG are respectively equal to the angles EGD and FGD. Also, by the third of Euclid's common notions, when equals are subtracted from equals, the remainders are themselves equal. It therefore follows that the angles EDF and EGF (coloured in yellow) are equal to one another.

As in the proof for case (ii), we now apply the SAS Congruence Rule (Euclid, *Elements*, I.4) to the triangles BAC and EDF. The sides BA and AC are equal to the sides ED and DF respectively. Moreover we have shown



that the angle BAC enclosed by the first pair of sides is equal to the angle EDF enclosed by the second pair of sides.



Applying the SAS Congruence Rule (Euclid, *Elements*, I.4), we conclude that the angles of the triangle BAC are equal to the corresponding angles of the triangle EDF and thus the angles ABC and DEF are equal to one another, and also the angles BCA and EFD are equal to one another. This completes the proof of Proposition 8 in case (iii) in which the straight line DG intersects the infinite straight line that passes through the points E and F as some intersection point that lies on the opposite side of the point F to the point E.

As noted before, the required result in cases (iv) and (v) follows on applying the resultd in cases (i) and (iii) with vertices appropriately relabelled. We have therefore completed the discussion of the alternative proof of Proposition 8 of Book I of Euclid's *Elements of Geometry* attributed to Philo of Byzantium.