Study Note—Euclid's *Elements*, Book I, Proposition 5

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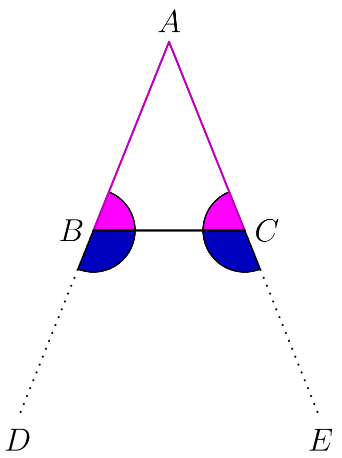
Let ABC be an isosceles triangle in which the sides AB and AC are equal to one another. Also let AB and AC be produced beyond B and C to points D and E respectively. Proposition 5 of Book I of Euclid's *Elements of Geometry* asserts, firstly, that the angles ABC and ACB at the base BC, opposite the equal sides, are equal to one another, and, secondly, that the angles DBC and ECB under the base are equal to one another.

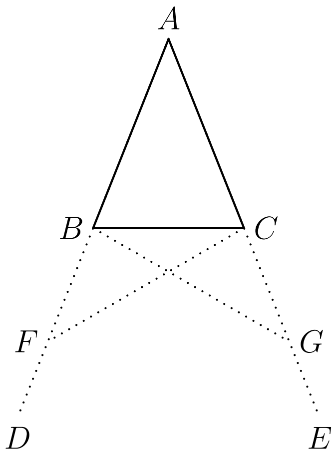


This proposition has acquired the nickname of the *Pons Asinorum*, or "Bridge of Asses".

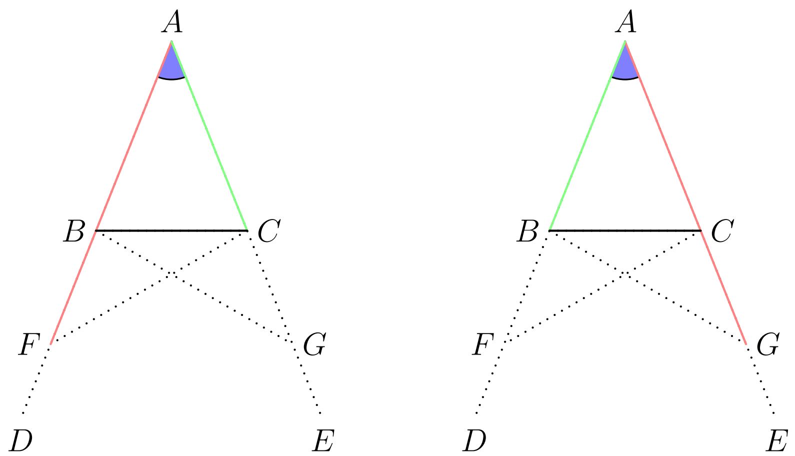
As commentators such as Proclus and Heath have observed, the result that the angles FBC and GCB under the base of the isosceles triangle are equal to one another is required in order to complete the proof of Proposition 7 of Book I in Euclid's *Elements* in some of the cases not explicitly considered by Euclid in the Greek text that has come down to us.

We now consider the proof of Proposition 5.

Euclid begins this proof by choosing a point F at random, somewhere past the point B, on the line BD that results on producing the side AB of the given isosceles triangle past B. Applying the result of Proposition 3 of Book I, one then locates a point G on the line CE that results on producing the side AC of the isosceles triangle beyond C, this point G being located so as to ensure that BF and CG are equal in length. Lines are then drawn from C to F and from B to G.

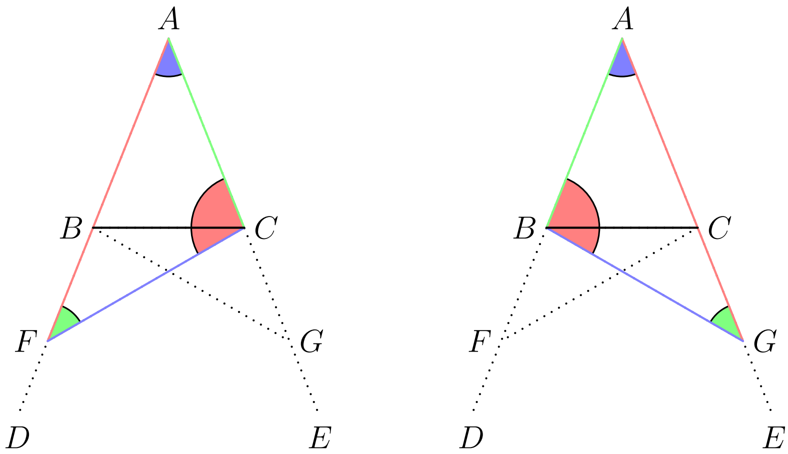


The next step is to apply the SAS Congruence rule to the triangles FAC and GAB. The sides FA and AC of the first triangle are equal in length to the sides GA and AB of the second triangle, respectively, and the angle BAC common to the two triangles.

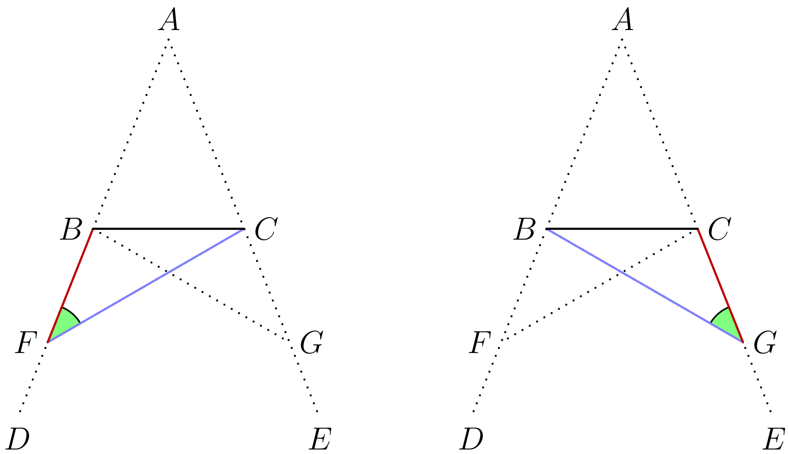


Applying the SAS Congruence Rule (Euclid, *Elements*, I.4), we conclude that the angles ACF and CFA are respectively equal to the angles ABG

and BGA, and also the finite lines FC and GB are equal to one another.

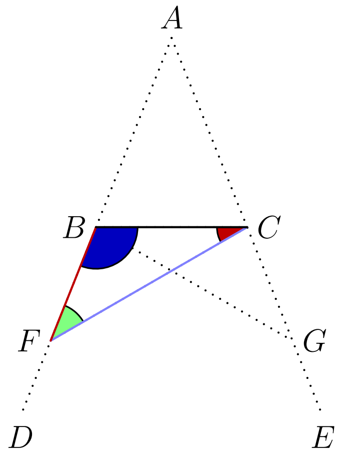


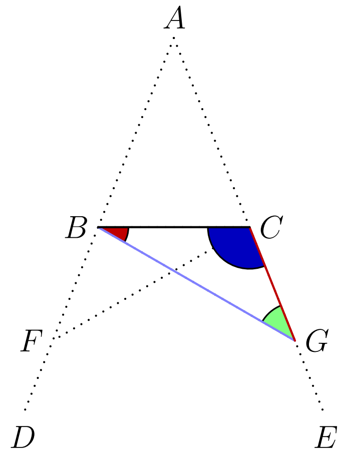
We now turn our attention to the triangles CFB and BGC. We have already established that the sides FB and FC of the first triangle are equal to the respective sides GC and GB of the second triangle. Also the angle BFC included between the specified sides of the first triangle is equal to the angle CGB included between the specified sides of the second triangle.



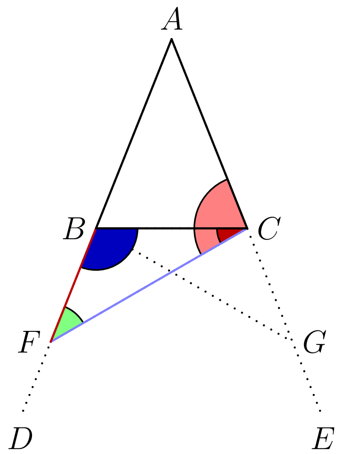
Applying the SAS Congruence Rule (Euclid, *Elements*, I.4), we conclude that the angles FBC and BCF of the first triangle are respectively equal to the angles GCB and CBG of the second triangle.

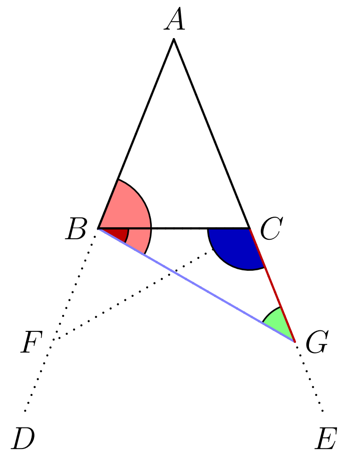
Now the angles FBC and GCB are identical to the angles DBC and ECB respectively. We have therefore established that the angles DBC and ECB under the base BC of the isosceles triangle ABC are equal to one another. This is the second of the two conclusions included in the statement of Proposition 5 that we were required to prove.



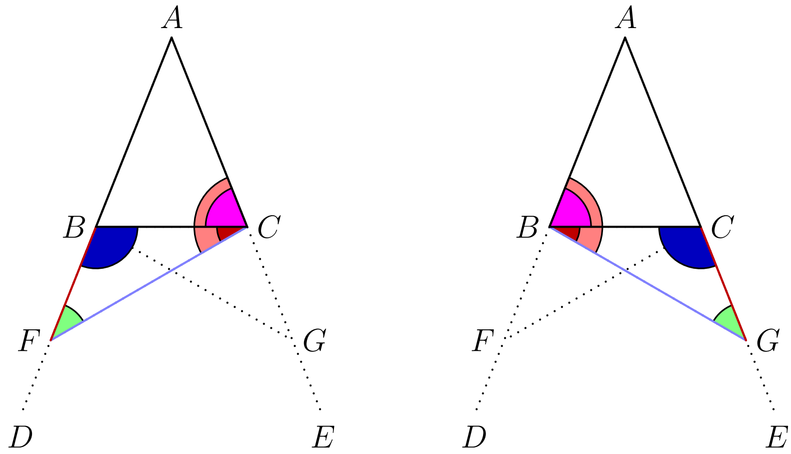


At this point we have also shown that the angles ACF and BCF are respectively equal to the angles ABG and CBG.





Subtracting the equal angles BCF and CBG from the equal angles ACF and ABG, we conclude that the angles ACB and ABC are equal to one another.



This completes this review of the proof of Proposition 5 of Book I of Euclid's *Elements of Geometry*.