ATTEMPT UP TO EIGHT QUESTIONS

1. (a) What is meant by saying that a topological space is compact?
   (b) Let $X$ and $Y$ be topological spaces, let $K$ be a compact subset of $Y$, and let $U$ be an open set in $X \times Y$. Let $V = \{x \in X : \{x\} \times K \subset U\}$. Prove that $V$ is an open set in $X$.
   (c) Prove that a Cartesian product of two compact topological spaces is compact.

2. (a) What is meant by saying that a topological space is connected? What is meant by saying that a topological space is path-connected?
   (b) Prove that a topological space $X$ is connected if and only if every continuous function $f : X \to \mathbb{Z}$ from $X$ to the set $\mathbb{Z}$ of integers is constant.
   (c) Prove that every path-connected topological space is connected.
   (d) Let $X$ be a non-empty connected open subset of $\mathbb{R}^n$ for some positive integer $n$. Prove that $X$ is path-connected. [Hint: use the connectedness of $X$ to show that, given a point $b$ of $X$, the set of points of $X$ that are endpoints of paths in $X$ starting at $b$ must be the whole of $X$.]

3. (a) Let $\hat{X}$ and $X$ be topological spaces, and let $p : \hat{X} \to X$ be a continuous map. What is meant by saying that an open set $U$ in $X$ is evenly covered by the map $p$? What is meant by saying that the map $p : \hat{X} \to X$ is a covering map?
   (b) Let $p : \hat{X} \to X$ be a covering map. Prove that $p(V)$ is open in $X$ for every open set $V$ in $\hat{X}$.
   (c) Let $p : \hat{X} \to X$ be a covering map, let $Z$ be a connected topological space, and let $g : Z \to \hat{X}$ and $h : Z \to \hat{X}$ be continuous maps. Suppose that $p \circ g = p \circ h$ and that $g(z) = h(z)$ for some $z \in Z$. Prove that $g = h$.
   (d) Let $H = \{z \in \mathbb{C} : \text{Re} z > 0\}$, and let $f : H \to \mathbb{C} \setminus \{0\}$ be defined by $f(z) = z^4$ for all $z \in H$. Is the map $f$ a covering map from $H$ to $\mathbb{C} \setminus \{0\}$? [Justify your answer.]
4. (a) Let \( \gamma: [0, 1] \to \mathbb{C} \) be a closed curve in the complex plane \( \mathbb{C} \), and let \( w \) be a complex number that does not lie on the curve \( \gamma \). Give the definition of the winding number \( n(\gamma, w) \) of the curve \( \gamma \) about \( w \).

(b) Let \( \gamma: [0, 1] \to \mathbb{C} \) be a closed curve in the complex plane that does not pass through zero, and let \( \eta: [0, 1] \to \mathbb{C} \) be the closed curve in the complex plane defined by \( \eta(t) = 1/\gamma(t) \) for all \( t \in [0, 1] \). Prove that \( n(\eta, 0) = -n(\gamma, 0) \).

(c) Let \( w \) be a complex number and, for each \( t \in [0, 1] \), let \( \gamma_t: [0, 1] \to \mathbb{C} \) be a closed curve in \( \mathbb{C} \) which does not pass through \( w \). Suppose that the map sending \((t, \tau) \in [0, 1] \times [0, 1] \) to \( \gamma_t(\tau) \) is a continuous map from \([0, 1] \times [0, 1] \) to \( \mathbb{C} \). Using the Monodromy Theorem, or otherwise, prove that \( n(\gamma_0, w) = n(\gamma_1, w) \).

(d) Let \( \gamma_0: [0, 1] \to \mathbb{C} \) and \( \gamma_1: [0, 1] \to \mathbb{C} \) be closed curves in \( \mathbb{C} \), and let \( w \) be a complex number which does not lie on the images of \( \gamma_0 \) and \( \gamma_1 \). Suppose that \( |\gamma_1(t) - \gamma_0(t)| < |w - \gamma_0(t)| \) for all \( t \in [0, 1] \). Prove that \( n(\gamma_0, w) = n(\gamma_1, w) \).

(e) State and prove the Fundamental Theorem of Algebra.

5. (a) What is meant by saying that a topological space \( X \) is simply-connected?

(b) Let \( X \) be a topological space, and let \( U \) and \( V \) be open subsets of \( X \) with \( U \cup V = X \). Suppose that \( U \) and \( V \) are simply-connected, and that \( U \cap V \) is non-empty and path-connected. Prove that \( X \) is simply-connected.

(c) Explain why the unit sphere \( S^n \) in \( \mathbb{R}^{n+1} \) is simply-connected when \( n > 1 \).

6. Prove that \( \pi_1(S^1, b) \cong \mathbb{Z} \), where \( b \) is a point on the circle \( S^1 \). [You may use, without proof, the Path Lifting Theorem and the Monodromy Theorem.]

7. (a) What is meant by saying that points \( v_0, v_1, \ldots, v_q \) in \( \mathbb{R}^k \) are geometrically independent (or affinely independent)?

(b) Define the concepts of simplex and simplicial complex. What is the polyhedron of a simplicial complex?

(c) An edge path \( v_0, v_1, \ldots, v_m \) joining vertices \( v_0 \) and \( v_m \) of a simplicial complex is a finite sequence of vertices such that \( v_{j-1} \) and \( v_j \) are endpoints of an edge of \( K \) for \( j = 1, 2, \ldots, m \). Prove that if the polyhedron \( |K| \) of a simplicial complex is a connected topological space then any two vertices of \( K \) can be joined by an edge path.

8. (a) Let \( K \) be a simplicial complex and let \( x \) be a point of the polyhedron \( |K| \) of \( K \). What is the star \( \text{st}_K(x) \) of \( x \) in \( K \)?

(b) Let \( K \) be a simplicial complex and let \( x \) be a point of the polyhedron \( |K| \) of \( K \). Prove that the star \( \text{st}_K(x) \) is an open subset of \( |K| \) and that \( x \in \text{st}_K(x) \). [You may use, without proof, the result that every point of \( |K| \) belongs to the interior of a unique simplex of \( K \).]

(c) Let \( K \) and \( L \) be simplicial complexes, and let \( f: |K| \to |L| \) be a continuous map. Prove that a function \( s: \text{Vert}K \to \text{Vert}L \) between the vertex sets of simplicial complexes \( K \) and \( L \) is a simplicial map, and is a simplicial approximation to \( f: |K| \to |L| \), if and only if \( f(\text{st}_K(v)) \subset \text{st}_L(s(v)) \) for all vertices \( v \) of \( K \).

(d) State and prove the Simplicial Approximation Theorem. [You may use, without proof, the result that the mesh of the \( j \)th barycentric subdivision of a simplicial complex \( K \) converges to zero as \( j \to +\infty \).]
9. (a) Let $C_q(K)$ be the $q$th chain group of a simplicial complex $K$, and let $\partial_q: C_q(K) \to C_{q-1}(K)$ be the boundary homomorphism. Write down the expression which defines $\partial_q(\langle v_0, v_1, \ldots, v_q \rangle)$ for an oriented $q$-simplex $\langle v_0, v_1, \ldots, v_q \rangle$ of $K$, and show that $\partial_q \circ \partial_{q+1} = 0$ when $2 \leq q \leq \dim K$.

(b) Define the group $Z_q(K)$ of $q$-cycles, the group $B_q(K)$ of $q$-boundaries and the $q$th homology group $H_q(K)$ of a simplicial complex $K$.

(c) Let $K$ be a simplicial complex. Suppose that there exists a vertex $w$ of $K$ with the following property: vertices $w, v_0, v_1, \ldots, v_q$ span a simplex of $K$ whenever $v_0, v_1, \ldots, v_q$ do so. Prove that $H_0(K) \cong \mathbb{Z}$ and $H_q(K) = 0$ for all $q > 0$.

10. Let $v_1, v_2, v_3$ be the vertices of a triangle contained in the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, and let $v_4 = (0, 0, 1)$ and $v_5 = (0, 0, -1)$. Let $K$ be the 2-dimensional simplicial complex consisting of the triangular faces, edges and vertices of the tetrahedra $\langle v_1v_2v_3v_4 \rangle$ and $\langle v_1v_2v_3v_5 \rangle$. (Thus the polyhedron of $|K|$ consists of the surfaces of the two tetrahedra which are glued together along a single common triangular face $\langle v_1v_2v_3 \rangle$.)

(a) Calculate the boundary of the 2-chain

$$a \langle v_1v_2v_4 \rangle + b \langle v_2v_3v_4 \rangle + c \langle v_3v_1v_4 \rangle + d \langle v_1v_2v_5 \rangle + e \langle v_2v_3v_5 \rangle + f \langle v_3v_1v_5 \rangle + g \langle v_1v_2v_3 \rangle$$

of $K$, where $a, b, c, d, e, f$ and $g$ are integers. Show that the group $Z_2(K)$ of 2-cycles of $K$ is given by $Z_2(K) = \{mz_1 + nz_2 : m, n \in \mathbb{Z}\}$, where

$$z_1 = \langle v_1v_2v_4 \rangle + \langle v_2v_3v_4 \rangle + \langle v_3v_1v_4 \rangle - \langle v_1v_2v_3 \rangle$$

$$z_2 = \langle v_1v_2v_5 \rangle + \langle v_2v_3v_5 \rangle + \langle v_3v_1v_5 \rangle - \langle v_1v_2v_3 \rangle,$$

and explain why $H_2(K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(b) For each of the following 1-chains of $K$ determine whether or not that 1-chain of $K$ is a 1-boundary of $K$ and, if so, find a 2-chain of $K$ of which it is the boundary:

(i) $\langle v_1v_2 \rangle + \langle v_2v_4 \rangle + \langle v_4v_3 \rangle + \langle v_3v_5 \rangle + \langle v_5v_1 \rangle$;

(ii) $2\langle v_1v_2 \rangle + 3\langle v_2v_3 \rangle + \langle v_3v_1 \rangle$.

11. (a) Define the following: an exact sequence of Abelian groups and homomorphisms; a chain complex; the homology groups of a chain complex; a chain map; a short exact sequence of chain complexes.

(b) Let $0 \to A \xrightarrow{p} B \xrightarrow{q} C \to 0$ be a short exact sequence of chain complexes. Prove that there is a well-defined homomorphism $\alpha: H_i(C) \to H_{i-1}(A)$ which sends the homology class $[z]$ of an element $z$ of $Z_i(C)$ to the homology class $[w]$ of any element $w$ of $Z_{i-1}(A)$ with the property that $p_{i-1}(w) = \partial_{i}(b)$ for some $b \in B$ satisfying $q_i(b) = z$. (Here $Z_i(C)$ and $Z_{i-1}(A)$ denote the kernels of the homomorphisms $\partial_i: C_i \to C_{i-1}$ and $\partial_{i-1}: A_{i-1} \to A_{i-2}$ respectively.)

12. Write an account of aspects of the theory of the topological classification of closed surfaces.

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