Course 421 Examination, June 1993: Worked Solutions

Structure of the examination

Question

1.	bookwork
2.	(a) definition (b) exercise (c) (i) problem set (ii) exercise
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4.	(a) definition (b) definition (c) exercise
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7.	(a) definition (b) definition (c) exercise
8.	(a) definition (b) bookwork (c) bookwork
9.	(a) definition (b) exercise (c) bookwork (d) bookwork
10.	(a) definition (b) exercise (c) exercise
11.	(a) definition (b) bookwork (c) problem set
12.	(a) problem set (b) problem set (c) exercise

where

'definition'= definition taken from lecture notes;
'bookwork'= result included in lecture notes;
'problem set'= problem taken from problem set;
'exercise'= not bookwork, and not included in any problem set.

Solutions

1. The *fundamental group* $\pi_1(X, x_0)$ is defined to be the set of equivalence classes of loops $\gamma\colon [0,1] \to X$ in X based at x_0 (i.e., satisfying $\gamma(0) = \gamma(1) = x_0$), where two such loops γ_0 and γ_1 are deemed to be equivalent if and only if $\gamma_0 \simeq \gamma_1 \operatorname{rel}\{0,1\}$ (i.e., there exists a homotopy $F\colon [0,1]\times[0,1] \to X$ with the properties that $F(t,0) = \gamma_0(t)$ and $F(t,1) = \gamma_1(t)$ for all $t \in [0,1]$ and $F(0,\tau) = F(1,\tau) = x_0$ for all $\tau \in [0,1]$). We denote the equivalence class of a loop γ by $[\gamma]$.

   The group operation on $\pi_1(X, x_0)$ is defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1.\gamma_2]$, where the product loop $\gamma_1.\gamma_2$ is defined by the formula

$$(\gamma_1.\gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

   First we show that the group operation on $\pi_1(X, x_0)$ is well-defined. Let γ_1, γ_1', γ_2 and γ_2' be loops in X based at the point x_0. Suppose that $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$. Let the map $F\colon [0,1]\times[0,1] \to X$ be defined by

$$F(t,\tau) = \begin{cases} F_1(2t,\tau) & \text{if } 0 \le t \le \frac{1}{2}, \\ F_2(2t-1,\tau) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

   where $F_1\colon [0,1]\times[0,1] \to X$ is a homotopy between γ_1 and γ_1',

2. between γ_2 and γ_2', and where the homotopies F_1 and F_2 map $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Then F is itself a homotopy from $\gamma_1.\gamma_2$ to $\gamma_1'.\gamma_2'$, and maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Thus $[\gamma_1.\gamma_2] = [\gamma_1'.\gamma_2']$, showing that the group operation on $\pi_1(X, x_0)$ is well-defined.

   Next we show that the group operation on $\pi_1(X, x_0)$ is assocative. Let γ_1, γ_2 and γ_3 be loops based at x_0, and let $\alpha = (\gamma_1.\gamma_2).\gamma_3$. Then $\gamma_1.(\gamma_2.\gamma_3) = \alpha \circ \eta$, where

$$\eta(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \le t \le \frac{1}{2}; \\ t - \frac{1}{4} & \text{if } \frac{1}{2} \le t \le \frac{3}{4}; \\ 2t - 1 & \text{if } \frac{3}{4} \le t \le 1. \end{cases}$$

   Thus the map $G\colon [0,1]\times[0,1] \to X$ defined by $G(t,\tau) = \alpha((1-\tau)t + \tau\theta(t))$ is a homotopy between $(\gamma_1.\gamma_2).\gamma_3$ and $\gamma_1.(\gamma_2.\gamma_3)$, and moreover this homotopy maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. It follows that $(\gamma_1.\gamma_2).\gamma_3 \simeq \gamma_1.(\gamma_2.\gamma_3) \operatorname{rel} \{0,1\}$ and hence $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$. This shows that the group operation on $\pi_1(X, x_0)$ is associative.

   Let $\varepsilon\colon [0,1] \to X$ denote the constant loop at x_0, defined by $\varepsilon(t) = x_0$ for all $t \in [0,1]$. Then $\varepsilon.\gamma = \gamma \circ \theta_0$ and $\gamma.\varepsilon = \gamma \circ \theta_1$ for any loop γ based at x_0, where

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{2}, \\ 2t-1 & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \qquad \theta_1(t) = \begin{cases} 2t & \text{if } 0 \le t \le \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

   for all $t \in [0,1]$. But the continuous map $(t,\tau) \mapsto \gamma(\tau\theta_j(t) + (1-\tau)t)$ is a homotopy between γ and $\gamma \circ \theta_j$ for $j = 0,1$ which sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\varepsilon.\gamma \simeq \gamma \simeq \gamma.\varepsilon \operatorname{rel} \{0,1\}$, and hence

$[\varepsilon][\gamma] = [\gamma] = [\gamma][\varepsilon]$. We conclude that $[\varepsilon]$ represents the identity element of $\pi_1(X, x_0)$.

It only remains to verify the existence of inverses. Given a loop $\gamma: [0,1] \to X$ based at x_0, we let $\gamma^{-1}(t) = \gamma(1-t)$ for all $t \in [0,1]$ We claim that $[\gamma^{-1}] = [\gamma]^{-1}$. Now the map $K: [0,1] \times [0,1] \to X$ defined by

$$K(t,\tau) = \begin{cases} \gamma(2\tau t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \gamma(2\tau(1-t)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

is a homotopy between the loops $\gamma.\gamma^{-1}$ and ε, and moreover this homotopy sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\gamma.\gamma^{-1} \simeq \varepsilon \operatorname{rel}\{0,1\}$, and thus $[\gamma][\gamma^{-1}] = [\gamma.\gamma^{-1}] = [\varepsilon]$. On replacing γ by γ^{-1}, we see also that $[\gamma^{-1}][\gamma] = [\varepsilon]$, and thus $[\gamma^{-1}] = [\gamma]^{-1}$, as required.

3. (a) An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p: \tilde{X} \to X$ is said to be a *covering map* if $p: \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) For each $x \in X$, let $\nu(x)$ be the number of elements in $p^{-1}(\{x\})$, when this set is finite, and let $\nu(x) = 0$ when $p^{-1}(\{x\})$ is infinite. If U is an evenly covered open set in X then $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$, each of which is mapped bijectively onto U under the map p. It follows that the function $\nu: X \to \mathbb{Z}$ is constant over each open set in X that is evenly covered by the map p. But each point of X belongs to some evenly-covered open set. It follows that $\nu: X \to \mathbb{Z}$ is continuous. But X is connected, by hypothesis. We deduce that $\nu: X \to \mathbb{Z}$ is constant. The value of this function is the required integer n.

(c)

(i) This is a covering map. The map is clearly surjective, and, given any point $\mathbf{v}$ of S^1, the set $S^1 \setminus \{\mathbf{v}\}$ is open in S^1 and is evenly covered by the map q. Indeed the preimage of this set has m connected components, each of which is mapped homeomorphically onto the given set.

(ii) This map is not a covering map. This can easily be deduced on applying (b). Indeed the preimage of $(0, 0)$ has two elements, whereas the preimage of $(1, 0)$ has only one element.

4. We regard S^1 as the unit circle in $\mathbb{R}^2$. Without loss of generality, we can take $b = (1, 0)$. Now the map $p: \mathbb{R} \to S^1$ which sends $t \in \mathbb{R}$ to $(\cos 2\pi t, \sin 2\pi t)$ is a covering map, and $b = p(0)$. Moreover $p(t_1) = p(t_2)$ if and only if $t_1 - t_2$ is an integer; in particular $p(t) = b$ if and only if t is an integer.

   Let α and β be loops in S^1 based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be paths in $\mathbb{R}$ that satisfy $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that α and β represent the same element of $\pi_1(S^1, b)$. Then there exists a homotopy $F: [0, 1] \times [0, 1] \to S^1$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all $t \in [0, 1]$, and $F(0, \tau) = F(1, \tau) = b$ for all $\tau \in [0, 1]$. It follows from the Monodromy Theorem that this homotopy lifts to a continuous map $G: [0, 1] \times [0, 1] \to \mathbb{R}$ satisfying $p \circ G = F$. Moreover $G(0, \tau)$ and $G(1, \tau)$ are integers for all $\tau \in [0, 1]$, since $p(G(0, \tau)) = b = p(G(1, \tau))$. Also $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$ are integers for all $t \in [0, 1]$, since $p(G(t, 0)) = \alpha(t) = p(\tilde{\alpha}(t))$ and $p(G(t, 1)) = \beta(t) = p(\tilde{\beta}(t))$. Now any continuous integer-valued function on $[0, 1]$ is constant, by the Intermediate Value Theorem. In particular the functions sending $\tau \in [0, 1]$ to $G(0, \tau)$ and $G(1, \tau)$ are constant, as are the functions sending $t \in [0, 1]$ to $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$. Thus

$$G(0, 0) = G(0, 1), \qquad G(1, 0) = G(1, 1),$$

$$G(1, 0) - \tilde{\alpha}(1) = G(0, 0) - \tilde{\alpha(0)}, \qquad G(1, 1) - \tilde{\beta}(1) = G(0, 1) - \tilde{\beta(0)}.$$

   On combining these results, we see that

$$\tilde{\alpha}(1) - \tilde{\alpha}(0) = G(1, 0) - G(0, 0) = G(1, 1) - G(0, 1) = \tilde{\beta}(1) - \tilde{\beta}(0).$$

   We conclude from this that there exists a well-defined function $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ characterized by the property that $\lambda([\alpha]) = \tilde{\alpha}(1) - \tilde{\alpha}(0)$ for all loops α based at b, where $\tilde{\alpha}: [0, 1] \to \mathbb{R}$ is any path in $\mathbb{R}$ satisfying $p \circ \tilde{\alpha} = \alpha$.

   Next we show that λ is a homomorphism. Let α and β be any loops based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β. The element $[\alpha][\beta]$ of $\pi_1(S^1, b)$ is represented by the product path $\alpha.\beta$, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

   Define a continuous path $\sigma: [0, 1] \to \mathbb{R}$ by

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \tilde{\beta}(2t - 1) + \tilde{\alpha}(1) - \tilde{\beta}(0) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

   (Note that $\sigma(t)$ is well-defined when $t = \frac{1}{2}$.) Then $p \circ \sigma = \alpha.\beta$ and thus

$$\lambda([\alpha][\beta]) = \lambda([\alpha.\beta]) = \sigma(1) - \sigma(0) = \tilde{\alpha}(1) - \tilde{\alpha}(0) + \tilde{\beta}(1) - \tilde{\beta}(0) = \lambda([\alpha]) + \lambda([\beta]).$$

   Thus $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ is a homomorphism.

   Now suppose that $\lambda([\alpha]) = \lambda([\beta])$. Let $F: [0, 1] \times [0, 1] \to S^1$ be the homotopy between α and β defined by

$$F(t, \tau) = p\left((1 - \tau)\tilde{\alpha}(t) + \tau\tilde{\beta}(t)\right),$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the lifts of α and β respectively starting at 0. Now $\tilde{\beta}(1) = \lambda([\beta]) = \lambda([\alpha]) = \tilde{\alpha}(1)$, and $\tilde{\beta}(0) = \tilde{\alpha}(0) = 0$. Therefore $F(0, \tau) = b = p(\tilde{\alpha}(1)) = F(1, \tau)$ for all $\tau \in [0, 1]$. Thus $\alpha \simeq \beta$ rel $\{0, 1\}$, and therefore $[\alpha] = [\beta]$. This shows that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is injective.

The homomorphism λ is surjective, since $n = \lambda([\gamma_n])$ for all $n \in \mathbb{Z}$, where the loop $\gamma_n\colon [0, 1] \to S^1$ is given by $\gamma_n(t) = p(nt) = (\cos 2\pi nt, \sin 2\pi nt)$ for all $t \in [0, 1]$. We conclude that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is an isomorphism, as required.

5. (a) The points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are said to be *geometrically independent* if the only solution of the linear system

$$\left\{ \begin{array}{rcl} \sum_{j=0}^q \lambda_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum_{j=0}^q \lambda_j & = & 0 \end{array} \right.$$

is the trivial solution $\lambda_0 = \lambda_1 = \cdots = \lambda_q = 0$.

(b) A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^q t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^q t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$.

A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

(i) if σ is a simplex belonging to K then every face of σ also belongs to K,

(ii) if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

(c) Let $\mathbf{x}, \mathbf{y} \in \sigma$. Then there exist real numbers $s_0, s_1, \ldots, s_q$ and $t_0, t_1, \ldots, t_q$, where $0 \leq s_j \leq 1$ and $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, and $\sum_{j=0}^q s_j = \sum_{j=0}^q t_j = 1$, such that $\sum_{j=0}^q s_j \mathbf{v}_j = \mathbf{x}$ and $\sum_{j=0}^q t_j \mathbf{v}_j = \mathbf{y}$. Let $u_j = (1-\lambda)s_j + \lambda t_j$. Then $\sum_{j=0}^q u_j \mathbf{v}_j = \lambda \mathbf{x} + (1-\lambda)\mathbf{y}$ and moreover $0 \leq u_j \leq 1$ for $j = 0, 1, \ldots, q$ (since the interval $[0,1]$ is convex) and $\sum_{j=0}^q u_j = (1-\lambda) + \lambda = 1$. It follows that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in \sigma$. We conclude that the simplex σ is convex.

Let K be a convex subset of $\mathbb{R}^k$ containing $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. We show, by induction on m, that K contains the m-face of σ_m of σ spanned by the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ for $m = 0, 1, \ldots, q$. The result is clearly true when $m = 0$, since $\sigma_0 = \{\mathbf{v}_0\}$. Suppose that it has been shown that $\sigma_{m-1} \subset K$. Let $\mathbf{x} \in \sigma_m$. Then there exist real numbers $t_0, t_1, \ldots, t_m$, where $0 \leq t_j \leq 1$ for $j = 0, 1 \ldots, m$ and $\sum_{j=0}^m t_j = 1$, such that $\mathbf{x} = \sum_{j=0}^m t_j \mathbf{v}_j$. But then $\mathbf{x} = (1-t_m)\mathbf{y} + t_m \mathbf{v}_m$, where

$$\mathbf{y} = \sum_{j=0}^{m-1} \frac{t_j}{1-t_m} \mathbf{v}_j.$$

Moreover $\mathbf{y} \in \sigma_{m-1}$, since $0 \leq t_j/(1-t_m) \leq 1$ for $j = 0, 1, \ldots, m-1$ and

$$\sum_{j=0}^{m-1} \frac{t_j}{1-t_m} = \frac{1}{1-t_m} \left(\sum_{j=0}^m t_j - t_m \right) = 1,$$

and $\sigma_{m-1} \subset K$. It follows that $\mathbf{y} \in K$. Also $\mathbf{v}_m \in K$, by hypothesis. It therefore follows from the convexity of K that $\mathbf{x} \in K$, since $0 \leq t_m \leq 1$. This shows that $\sigma_m \subset K$ for all m between 0 and q. In particular $\sigma \subset K$, since $\sigma = \sigma_q$, as required

6. (a) A *Sperner labelling* of the vertices of K is a function, labelling each vertex of $\mathbf{v}$ with an integer between 0 and n, with the following properties:—

   (i) for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,

   (ii) if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

(b) *Sperner's Lemma.* Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.

Proof. Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occuring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$. Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2\big(N(0, 1, \ldots, n-1) - M\big),$$

since any $(n-1)$-simplex of K that is contained in a face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required.

(c) Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s: K \to L$ is a simplicial approximation to $r: \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r: \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

7. Let our exchange economy consist of n commodities and m households. We suppose that household h is provided with an initial endowment $\overline{x}_{hi}$ of commodity i, where $\overline{x}_{hi} \geq 0$. Thus the initial endowment of household h can be represented by a vector $\overline{\mathbf{x}}_h$ in $\mathbb{R}^n$ whose ith component is $\overline{x}_{hi}$. The prices of the commodities are given by a price vector $\mathbf{p}$ whose ith component p_i specifies the price of a unit of the ith commodity: a price vector $\mathbf{p}$ is required to satisfy $p_i \geq 0$ for all i. Then the value of the initial endowment of household h at the given prices is $\mathbf{p}.\overline{\mathbf{x}}_h$. Let $x_{hi}(\mathbf{p})$ be the quantity of commodity i that household h seeks to purchase at prices $\mathbf{p}$, and let $\mathbf{x}_h(\mathbf{p}) \in \mathbb{R}^n$ be the vector whose ith component is $x_{hi}(\mathbf{p})$. The budget constraint certainly ensures that $\mathbf{p}.(\mathbf{x}_h(\mathbf{p}) - \overline{\mathbf{x}}_h) \leq 0$ (i.e., the value of the goods purchased cannot exceed the value of the initial endowment at the given prices). We assume that the value of the commodities that each household seeks to purchase is equal to the value of its initial endowment, and thus $\mathbf{p}.\mathbf{x}_h(\mathbf{p}) = \mathbf{p}.\overline{\mathbf{x}}_h$. Also the preferences of the household will only depend on the relative prices of the commodities, and therefore $\mathbf{x}_h(\lambda\mathbf{p}) = \mathbf{x}_h(\mathbf{p})$ for all $\lambda > 0$.

   Now the total supply of each commodity in the economy is represented by the vector $\sum_h \overline{\mathbf{x}}_h$, and the total demand at prices $\mathbf{p}$ is represented by $\sum_h \mathbf{x}_h(\mathbf{p})$. The *excess demand* in the economy at prices $\mathbf{p}$ is therefore represented by the vector $\mathbf{z}(\mathbf{p})$, where $\mathbf{z}(\mathbf{p}) = \sum_h (\mathbf{x}_h(\mathbf{p}) - \overline{\mathbf{x}}_h)$. Let $z_i(\mathbf{p})$ be the ith component of $\mathbf{z}(\mathbf{p})$. Then $z_i(\mathbf{p}) > 0$ when the demand for the ith commodity exceeds supply, whereas $z_i(\mathbf{p}) < 0$ when the supply exceeds demand. Note that $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$ for any price vector $\mathbf{p}$. This identity, known as *Walras' Law*, follows immediately on summing the budget constraint $\mathbf{p}.\mathbf{x}_h(\mathbf{p}) = \mathbf{p}.\overline{\mathbf{x}}_h$ over all households.

   Consider an exchange economy consisting of a finite number of infinitely divisible commodities and a finite number of households. Let the excess demand in the economy at prices $\mathbf{p}$ *be given by* $\mathbf{z}(\mathbf{p})$*, where*

   (i) the excess demand vector $\mathbf{z}(\mathbf{p})$ *is well-defined for any price vector* $\mathbf{p}$*, and depends continuously on* $\mathbf{p}$*,*

   (ii) $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$ *for any price vector* $\mathbf{p}$ *(Walras' Law).*

   Then there exist equilibrium prices $\mathbf{p}^*$ *at which* $z_i(\mathbf{p}^*) \leq 0$ *for all* i*.*

   Proof. Let Δ be the $(n-1)$-dimensional simplex in $\mathbb{R}^n$ consisting of all points $(p_1, p_2, \ldots, p_n)$ in $\mathbb{R}^n$ satisfying $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n p_i = 1$, and let $\mathbf{v}\colon \Delta \to \mathbb{R}^n$ be the function with ith component v_i given by

$$v_i(\mathbf{p}) = \begin{cases} p_i + z_i(\mathbf{p}) & \text{if } z_i(\mathbf{p}) > 0; \\ p_i & \text{if } z_i(\mathbf{p}) \leq 0. \end{cases}$$

   Note that $\mathbf{v}(\mathbf{p}) \neq \mathbf{0}$ and the components of $\mathbf{v}(\mathbf{p})$ are non-negative for all $\mathbf{p} \in \Delta$. It follows that there is a well-defined map $\varphi\colon \Delta \to \Delta$ given by

$$\varphi(\mathbf{p}) = \frac{1}{\sum\limits_{i=1}^{n} v_i(\mathbf{p})} \mathbf{v}(\mathbf{p}),$$

The Brouwer Fixed Point Theorem ensures that there exists $\mathbf{p}^* \in \Delta$ satisfying $\varphi(\mathbf{p}^*) = \mathbf{p}^*$. Then $\mathbf{v}(\mathbf{p}^*) = \lambda\mathbf{p}^*$ for some $\lambda \geq 1$. We claim that $\lambda = 1$.

Suppose that it were the case that $\lambda > 1$. Then $v_i(\mathbf{p}^*) > p_i^*$, and thus $z_i(\mathbf{p}^*) > 0$ whenever $p_i^* > 0$. But $p_i^* \geq 0$ for all i, and $p_i^* > 0$ for at least one value of i, since $\mathbf{p}^* \in \Delta$. It would follow that $\mathbf{p}^*.\mathbf{z}(\mathbf{p}^*) > 0$, contradicting Walras' Law. We conclude that $\lambda = 1$, and thus $v_i = p_i^*$ and $z_i(\mathbf{p}^*) \leq 0$ for all i, as required.

Note that if $z_i(\mathbf{p}^*) \leq 0$ for all i and $\mathbf{p}^*.\mathbf{z}(\mathbf{p}^*) = 0$, then $p_i^* z_i(\mathbf{p}^*) \leq 0$ for all i, and $z_i(\mathbf{p}^*) = 0$ whenever $p_i > 0$. Thus, at equilibrium prices, supply always equals or exceeds demand, and supply equals demand for those commodities with positive prices.

8. (a) $\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle) = \sum_{j=0}^{q}(-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle$,

where $\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle = \langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_q\rangle$.

(b) $Z_q(K) = \ker(\partial_q\colon C_q(K) \to C_{q-1}(K))$, $B_q(K) = \partial_{q+1}(C_q(K))$, $H_q(K) = Z_q(K)/B_q(K)$.

(c) $\partial_2(\alpha) = (n_2 + n_3)\rho_{01} + (n_1 - n_3)\rho_{02} - (n_1 + n_2)\rho_{03} + (n_0 + n_3)\rho_{12} + (n_2 - n_0)\rho_{13} + (n_0 + n_1)\rho_{23}$. Therefore $\partial_2(\alpha) = 0$ if and only if $n_0 = -n_1 = n_2 = -n_3$.

$$\begin{aligned}\partial_1(\beta) \;=\; & -(m_{01} + m_{02} + m_{03})\langle P_0\rangle + (m_{01} - m_{12} - m_{13})\langle P_1\rangle \\ & + (m_{02} + m_{12} - m_{23})\langle P_2\rangle + (m_{03} + m_{13} + m_{23})\langle P_3\rangle.\end{aligned}$$

Therefore $\partial_1(\beta) = 0$ if and only if

$$m_{01} + m_{02} + m_{03} = 0, \quad m_{01} - m_{12} - m_{13} = 0,$$

$$m_{02} + m_{12} - m_{23} = 0, \quad m_{03} + m_{13} + m_{23} = 0.$$

Substituting for m_{ij} in terms of n_i shows that $\partial_1(\partial_2(\alpha)) = 0$. Conversely if $\partial_1(\beta) = 0$ then $\beta = \partial_2(\alpha)$, where the coefficients n_i of α are given in terms of the coeffients m_{ij} of β by the formulae

$$n_0 = 0, \quad n_1 = m_{23}, \quad n_2 = m_{13}, \quad n_3 = m_{12}.$$

Observe also that the coefficients of $\langle P_i\rangle$ in $\partial_1(\beta)$ sum to zero, and, conversely,

$$j_0\langle P_0\rangle + j_1\langle P_1\rangle + j_2\langle P_2\rangle + j_2\langle P_3\rangle = \partial_1\left(j_1\rho_{01} + j_2\rho_{02} + j_3\rho_{03}\right) \text{ when } j_0 + j_1 + j_2 + j_3 = 0.$$

Putting all these results together, we see that

$$\begin{aligned}Z_2(K) &= \{n(\tau_0 - \tau_1 + \tau_2 - \tau_3)\}, \\ B_2(K) &= 0, \\ Z_1(K) &= \{(n_2 + n_3)\rho_{01} + (n_1 - n_3)\rho_{02} - (n_1 + n_2)\rho_{03} + n_1\rho_{23} + n_2\rho_{13} + n_3\rho_{12}\}, \\ B_1(K) &= Z_1(K), \\ Z_0(K) &= C_0(K) = \{j_0\langle P_0\rangle + j_1\langle P_1\rangle + j_2\langle P_2\rangle + j_3\langle P_3\rangle\}, \\ B_0(K) &= \{j_0\langle P_0\rangle + j_1\langle P_1\rangle + j_2\langle P_2\rangle + j_3\langle P_3\rangle\}.\end{aligned}$$

where the coefficients n_i and j_i are integers. It follows that $H_2(K) = Z_2(K) \cong \mathbb{Z}$, $H_1(K) = 0$, and $H_0(K) = C_0(K)/B_0(K) \cong \mathbb{Z}$. Indeed the function sending the homology class of $j_0\langle P_0\rangle + j_1\langle P_1\rangle + j_2\langle P_2\rangle + j_3\langle P_3\rangle$ to $j_0 + j_1 + j_2 + j_3$ is an isomorphism from $H_0(K)$ to $\mathbb{Z}$.

9. (a) Vertices $\mathbf{y}$ and $\mathbf{z}$ of K can be joined by an edge path if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of K with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to K for $j = 1, 2, \ldots, m$.

(b) It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required.

(c) Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be the vertices of the simplicial complex K. Every 0-chain of K can be expressed uniquely as a formal sum of the form

$$n_1\langle \mathbf{u}_1 \rangle + n_2\langle \mathbf{u}_2 \rangle + \cdots + n_r\langle \mathbf{u}_r \rangle$$

for some integers $n_1, n_2, \ldots, n_r$. There is therefore a well-defined homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ defined by

$$\varepsilon\left(n_1\langle \mathbf{u}_1 \rangle + n_2\langle \mathbf{u}_2 \rangle + \cdots + n_r\langle \mathbf{u}_r \rangle\right) = n_1 + n_2 + \cdots + n_r.$$

Now $\varepsilon(\partial_1(\langle \mathbf{y}, \mathbf{z} \rangle)) = \varepsilon(\langle \mathbf{z} \rangle - \langle \mathbf{y} \rangle) = 0$ whenever $\mathbf{y}$ and $\mathbf{z}$ are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0$, and hence $B_0(K) \subset \ker \varepsilon$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ be vertices of K determining an edge path. Then

$$\langle \mathbf{v}_m \rangle - \langle \mathbf{v}_0 \rangle = \partial_1 \left(\sum_{j=1}^{m} \langle \mathbf{v}_{j-1}, \mathbf{v}_j \rangle \right) \in B_0(K).$$

Now K is connected, and therefore any pair of vertices of K can be joined by an edge path. We deduce that $\langle \mathbf{z} \rangle - \langle \mathbf{y} \rangle \in B_0(K)$ for all vertices $\mathbf{y}$ and $\mathbf{z}$ of K. Thus if $c \in \ker \varepsilon$, where $c = \sum_{j=1}^{r} n_j \langle \mathbf{u}_j \rangle$, then $\sum_{j=1}^{r} n_j = 0$, and hence $c = \sum_{j=2}^{r} n_j(\langle \mathbf{u}_j \rangle - \langle \mathbf{u}_1 \rangle)$. But $(\langle \mathbf{u}_j \rangle - \langle \mathbf{u}_1 \rangle) \in B_0(K)$. It follows that $c \in B_0(K)$. Thus We conclude that that $\ker \varepsilon \subset B_0(K)$, and hence $\ker \varepsilon = B_0(K)$.

Now the homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ is surjective and its kernel is $B_0(K)$. It follows from this that it induces an isomorphism from

$C_0(K)/B_0(K)$ to $\mathbb{Z}$. However $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition). Thus $H_0(K) \equiv C_0(K)/B_0(K) \cong \mathbb{Z}$, as required.

10. (a) A sequence of groups and homomorphisms is said to be *exact* if the image of each homomorphism is the kernel of the succeeding homomorphism in the sequence.

(b) Let $z \in H$. Then there exists $x \in G$ satisfying $\beta(x) = z$, since $\beta\colon G \to H$ is surjective. If x and y are elements of G satisfying $\beta(x) = z$ and $\beta(y) = z$ then $\beta(x - y) = 0$, hence $x - y = \alpha(w)$ for some $w \in F$. But then $\theta(x - y) = \theta(\alpha(w)) = 0$, and hence $\theta(x) = \theta(y)$. It follows that there is a well-defined element $\varphi(z)$ of K such that $\theta(x) = \varphi(z)$ for all $x \in G$ satisfying $\beta(x) = z$.

We must show that $\varphi\colon H \to K$ is a homomorphism. Let z_1 and z_2 be elements of H. Then $\varphi(z_1) = \theta(x_1)$ and $\varphi(z_2) = \theta(x_2)$, where $x_1, x_2 \in G$ are chosen to satisfy $\beta(x_1) = z_1$ and $\beta(x_2) = z_2$. But then $\theta(x_1 + x_2) = \theta(x_1) + \theta(x_2)$ and $\beta(x_1 + x_2) = z_1 + z_2$. It follows that $\varphi(z_1 + z_2) = \varphi(z_1) + \varphi(z_2)$. Thus $\varphi\colon H \to K$ is indeed a homomorphism, and $\varphi \circ \beta = \theta$. Moreover the homomorphism φ is uniquely determined since $\beta(G) = H$.

(c) Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2(\theta_1(w)) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

(d) Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b),$$

hence $\phi_3(a - \psi_3(c)) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then

$$\psi_3(\theta_2(e)) = \phi_2(\psi_2(e)) = a - \psi_3(c).$$

Hence $a = \psi_3(c + \theta_2(e))$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism, as required.

11. (a) Let

$$i_q: C_q(L \cap M) \to C_q(L), \qquad j_q: C_q(L \cap M) \to C_q(M),$$

$$u_q: C_q(L) \to C_q(K), \qquad v_q: C_q(M) \to C_q(K)$$

be the inclusion homomorphisms induced by the inclusion maps $i: L \cap M \hookrightarrow L$, $j: L \cap M \hookrightarrow M$, $u: L \hookrightarrow K$ and $v: M \hookrightarrow K$. The *Mayer-Vietoris* exact sequence is the sequence

$$\cdots \xrightarrow{\alpha_{q+1}} H_q(L \cap M) \xrightarrow{k_*} H_q(L) \oplus H_q(M) \xrightarrow{w_*} H_q(K) \xrightarrow{\alpha_q} H_{q-1}(L \cap M) \xrightarrow{k_*} \cdots,$$

of homology groups, where

$$k_*(\beta) = (i_*(\beta), -j_*(\beta)), \qquad w_*(\beta', \beta'') = u_*(\beta') + v_*(\beta''),$$

and $\alpha_q([z]) = [\partial_q(c')]$ for any $z \in Z_q(K)$, where $c' \in C_q(L)$ and $c'' \in C_q(M)$ satisfy $c' + c'' = z$.

(b) The homomorphism $k: G \to \mathbb{Z}$ is surjective, by exactness, hence there exists $g_0 \in G$ satisfying $k(g_0) = 1$. Let $s(n) = ng_0$ (where ng_0 is the sum of n copies of g_0). Then $s: \mathbb{Z} \to G$ is a well-defined homomorphism, and $k(s(n)) = k(ng_0) = nk(g_0) = n$ for all $n \in \mathbb{Z}$.

Let $q(m, n) = h(m) + s(n)$ for all $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$. If $q(m, n) = 0$ then

$$0 = k(q(m, n)) = k(h(m)) + k(s(n)) = n,$$

since $k \circ h = 0$ (by exactness) and $k \circ s$ is the identity homomorphism. But then $0 = q(m, n) = h(m)$ and $h: \mathbb{Z} \to G$ is injective, hence $m = 0$. Thus $m = n = 0$ whenever $q(m, n) = 0$. This shows that $q: \mathbb{Z} \oplus \mathbb{Z} \to G$ is injective.

Let $g \in G$. We must show that there exists $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ satisfying $q(m, n) = g$. Let $n = k(g)$. Then $k(g - s(n)) = k(g) - n = 0$, hence $g - s(n) = h(m)$ for some $m \in \mathbb{Z}$, by exactness. But then $g = q(m, n)$. Therefore $q: \mathbb{Z} \oplus \mathbb{Z} \to G$ is injective, and is thus an isomorphism, as required.

(c) The union of the 2-sphere and the given disk is homeomorphic to the polyhedron of some simplicial complex K which is the union of subcomplexes L and M, where L is homeomorphic to the disk, M is homeomorphic to the 2-sphere, and $L \cap M$ is homeomorphic to the unit circle. Then

$$H_0(L \cap M) \cong H_0(L) \cong H_0(M) \cong \mathbb{Z},$$

$$H_1(L) = 0, \quad H_1(M) = 0, \quad H_1(L \cap M) \cong \mathbb{Z},$$

$$H_2(L) = 0, \quad H_2(M) \cong \mathbb{Z}, \quad H_2(L \cap M) = 0.$$

Now $H_0(K) \cong \mathbb{Z}$, since $|K|$ is connected.

Next note that the homomorphisms $i_*: H_0(L \cap M) \to H_0(L)$ and $j_*: H_0(L \cap M) \to H_0(M)$ are isomorphisms, hence $k_*: H_0(L \cap M) \to H_0(L) \oplus H_0(M)$ is injective. It follows from the exactness of the Mayer-Vietoris sequence that $\alpha_1: H_1(K) \to H_0(L \cap M)$ is the zero homomorphism, and hence $w_*\colon H_1(L) \oplus H_1(M) \to H_1(K)$ is surjective. But $H_1(L) = 0$ and $H_1(M) = 0$. It follows that $H_1(K) = 0$.

Using the exactness of the Mayer-Vietoris sequence, and the facts that

$$H_2(L \cap M) = 0, \quad H_2(L) = 0, \qquad H_1(L) = 0, \qquad H_1(M) = 0,$$

we see that

$$0 \longrightarrow H_2(M) \xrightarrow{v_*} H_2(K) \xrightarrow{\alpha_2} H_1(L \cap M) \longrightarrow 0$$

is exact. But $H_2(M) \cong \mathbb{Z}$ and $H_1(L \cap M) \cong \mathbb{Z}$. It follows from (c) that $H_2(K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Finally we note that $H_q(K) = 0$ for $q > 2$, since $\dim K = 2$. We have thus shown that

$$H_0(K) \cong \mathbb{Z}, \quad H_1(K) = 0, \quad H_2(K) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

This completes the calculation of the homology groups of K.

12. (a) Two simplicial maps $s\colon K \to L$ and $t\colon K \to L$ are said to be *contiguous* if, given any simplex σ of K, there exists a simplex τ of L such that $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ.

(b) Let $\mathbf{x}$ be a point in the interior of some simplex σ of K. Then $f(\mathbf{x})$ belongs to the interior of a unique simplex τ of L, and moreover $s(\mathbf{x}) \in \tau$ and $t(\mathbf{x}) \in \tau$, since s and t are simplicial approximations to the map f. But $s(\mathbf{x})$ and $t(\mathbf{x})$ are contained in the interior of the simplices $s(\sigma)$ and $t(\sigma)$ of L. It follows that $s(\sigma)$ and $t(\sigma)$ are faces of τ, and hence $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ, as required.

(c) $D_1(\langle \mathbf{v}\rangle) = \langle s(\mathbf{v}), t(\mathbf{v})\rangle$ for each vertex $\mathbf{v}$ of K, hence $\partial_1(D_1(\langle \mathbf{v}\rangle)) = \langle t(\mathbf{v})\rangle - \langle s(\mathbf{v})\rangle$. Thus $\partial_1 \circ D_0 = t_0 - s_0$.

Now let $q > 0$. Then

$$
\begin{aligned}
&(\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle)))\\
&\qquad = \partial_{q+1}\left(\sum_{j=0}^{q}(-1)^j \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle\right)\\
&\qquad = \sum_{k<j}(-1)^{j+k}\langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_k)}, \ldots s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle\\
&\qquad\qquad + \sum_{j=0}^{q}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_{j-1}), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle\\
&\qquad\qquad - \sum_{j=0}^{q}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_{j+1}), \ldots, t(\mathbf{v}_q)\rangle\\
&\qquad\qquad + \sum_{k>j}(-1)^{j+k+1}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_k)}, \ldots, t(\mathbf{v}_q)\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&(D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle)))\\
&\qquad = D_{q-1}\left(\sum_{k=0}^{q}(-1)^k \mathbf{v}_0, \ldots \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q\rangle\right)\\
&\qquad = \sum_{j<k}(-1)^{j+k}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_k)}, \ldots, t(\mathbf{v}_q)\rangle\\
&\qquad\qquad + \sum_{j>k}(-1)^{j+k-1}\langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_k)}, \ldots s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle,
\end{aligned}
$$

hence

$$
\begin{aligned}
&(\partial_{q+1} \circ D_q + D_{q-1} \circ \partial_q)(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle)\\
&\qquad = \sum_{j=0}^{q}\Big(\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_{j-1}), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle\\
&\qquad\qquad - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_{j+1}), \ldots, t(\mathbf{v}_q)\rangle\Big)\\
&\qquad = \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q)\rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q)\rangle.
\end{aligned}
$$

Thus $\partial_{q+1} \circ D_q + D_{q-1} \circ \partial_q = t_q - s_q$ for all $q > 0$. It follows that $t_q(z) - s_q(z) = \partial_{q+1}(D_q(z))$ for any q-cycle z of K, and therefore $s_*([z]) = t_*([z])$. Thus $s_* = t_*$ as homomorphisms from $H_q(K)$ to $H_q(L)$, as required.

13. (a) Let $f: S^1 \to M$ be defined by

$$f(\cos 2\pi s, \sin 2\pi s) = q(s, \tfrac{1}{2}),$$

where $q: [0,1] \times [0,1] \to M$ is the identification map, and let $g: M \to S^1$ be defined such that

$$g(q(s,t)) = (\cos 2\pi s, \sin 2\pi s).$$

Then $g \circ f$ is the identity map of S^1, and $f \circ g$ is homotopic to the identity map of M by means of the the homotopy

$$q((s,t),\tau)) \mapsto q(s, (1-t)\tau + \tfrac{1}{2}\tau).$$

It follows that $f: S^1 \to M$ induces isomorphisms of homology groups. Indeed $g_*: H_q(M) \to H_q(S^1)$ is the inverse of $f_*: H_q(S^1) \to H_q(M)$. It follows that $H_0(M) \cong \mathbb{Z}$, $H_1(M) \cong \mathbb{Z}$ and $H_q(M) = 0$ for all $q \geq 2$.

(b) The boundary ∂M of M is homeomorphic to S^1, and the map $g \circ i: \partial M \to S^1$ has winding number 2. It follows that $(g \circ i)_*([w]) = 2([z_0])$, where $[z_0]$ is some generator of $H_1(S^1)$. Thus $i_*([w]) = 2[z]$, where $[z] = f_*[z_0]$. Moreover the generators of $H_1(M)$ are $\pm[z]$. The result follows.

(c) Let us write $\mathbb{R}P^2 = M \cup D$, where $\mathbb{R}P^2$ denotes the real projective plane, M is the Möbius strip, D is the closed disk. Then $\partial M = M \cap D = \partial D$, $M \cap D$ is homeomorphic to S^1, and the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\alpha_{q+1}} H_q(M \cap D) \xrightarrow{k_*} H_q(M) \oplus H_q(D) \xrightarrow{w_*} H_q(\mathbb{R}P^2) \xrightarrow{\alpha_q} H_{q-1}(M \cap D) \xrightarrow{k_*} \cdots$$

is exact.

Now $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$, since $\mathbb{R}P^2$ is connected. Also $H_q(\mathbb{R}P^2) = 0$ if $q < 0$ or $q > 2$, since the real projective plane is triangulated by a simplicial complex of dimension 2.

Now the components of $k_*: H_0(M \cap D) \to H_0(M) \oplus H_0(D)$ are isomorphisms, since $M \cap D$, M and D are connected. Therefore this homomorphism is injective. It follows from exactness that $\alpha_1: H_1(\mathbb{R}P^2) \to H_0(M \cap D)$ is the zero homomorphism. Also $H_1(D) = 0$. It follows that the sequence

$$H_1(\partial M) \xrightarrow{i_*} H_1(M) \longrightarrow H_1(\mathbb{R}P^2)$$

is exact. It follows from (b) that

$$H_1(\mathbb{R}P^2) \cong H_1(M)/i_*(H_1(\partial M)) \cong \mathbb{Z}_2.$$

Also the sequence

$$0 \longrightarrow H_2(\mathbb{R}P^2) \longrightarrow H_1(\partial M) \xrightarrow{i_*} H_1(M)$$

is exact, and therefore

$$H_2(\mathbb{R}P^2) \cong \ker(i_*: H_1(\partial M) \to H_1(M)).$$

But it follows from (b) that $i_*: H_1(\partial M) \to H_1(M)$ is injective. It follows that $H_2(\mathbb{R}P^2) = 0$. This completes the calculation of the homology groups of $\mathbb{R}P^2$.