Course 421: Annual Examination, worked solutions

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Provisional marking scheme

Each question is marked out of 20. The marks of the best six questions are then added together. The resultant mark is then either converted to a percentage as it stands, or else a piecewise linear transformation of the mark will be applied, if this is considered appropriate after reviewing all the marked scripts.

1. (a) 3 marks, (b) 3 marks, (c) 4 marks (d) 6 marks (e) 4 marks
2. (a) 4 marks, (b) 8 marks, (c) 4 marks (d) 4 marks
3. (a) 6 marks, (b) 10 marks, (c) 4 marks
4. 8 marks for showing the existence of a well-defined function $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$; 4 marks for showing that it is a homomorphism, 4 marks for showing that it is injective, 4 marks for showing that it is surjective.
5. (a) 3 marks, (b) 10 marks, (c) 7 marks
6. (a) 6 marks, (b) 4 marks, (c) 10 marks
7. (a) 10 marks, (b) 5 marks, (c) 5 marks
8. (a) 6 marks, (b) 14 marks
9. (a) 6 marks, (b) 14 marks

Style of questions

1. (a)–(d) bookwork, (e) not bookwork
2. (a)–(c) bookwork, (d) not bookwork
3. (a) bookwork, (b) and (c) not bookwork
4. bookwork
5. bookwork
6. bookwork
7. not bookwork—problems of this general type have been discussed in class
8. bookwork
9. partially bookwork—the homology groups of torus, Klein bottle and real projective plane have been discussed in class, but presented somewhat differently. (Essentially, I gave a simultaneous treatment of the torus, Klein bottle and real projective plane.)

1. (a) Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

(b) Let $\mathcal{U}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. It follows that A is compact, as required.

(c) Let p be a point of X that does not belong to A, and let $f(x) = d(x,p)$, where d is the distance function on X. It follows that there is a point q of A such that $f(a) \geq f(q)$ for all $a \in A$, since A is compact. Now $f(q) > 0$, since $q \neq p$. Let δ satisfy $0 < \delta \leq f(q)$. Then the open ball of radius δ about the point p is contained in the complement of A, since $f(x) < f(q)$ for all points x of this open ball. It follows that the complement of A is an open set in X, and thus A itself is closed in X.

(d) Suppose that K is compact. Then K is closed, since $\mathbb{R}^n$ is a metric space, and a compact subset of a metric space is closed. For each natural number m, let B_m be the open ball of radius m about the origin, given by $B_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}$. Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that $K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}$. But then $K \subset B_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$, and thus K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem, and C is the Cartesian product of n copies of the compact set $[-L, L]$. It follows from C is a finite product of compact spaces, and is therefore compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact. Thus K is compact, as required.

(e) Let $\mathbf{x}$ be a point of $\mathbb{R}^n \setminus K_1$, and let $f: K \to \mathbb{R}$ be defined by $f(\mathbf{y}) = |\mathbf{y} \setminus \mathbf{x}|$. Then $f(\mathbf{y}) > 1$ for all points $\mathbf{y}$ of K. Now $f(K)$ is a compact subset of $\mathbb{R}$ (since continuous functions map compact sets to compact sets), and therefore $f(K)$ is a closed set in $\mathbb{R}$. Also $f(K) \cap [0, 1] = \emptyset$. It follows that there exists $\delta > 0$ such that $f(K) \cap [0, 1 + \delta) = \emptyset$. But then Then $|\mathbf{y} - \mathbf{x}| > 1 + \delta$ for all points $\mathbf{y}$ of K, and therefore the open ball of radius δ about $\mathbf{x}$ is contained $\mathbb{R}^n \setminus K_1$. This shows that $\mathbb{R}^n \setminus K_1$ is open, and therefore K_1 is closed in $\mathbb{R}^n$. Moreover K_1 is clearly bounded, for the compact set K is wholly contained within some ball of radius R about the origin, and K_1 is then contained within the ball of radius $R + 1$ about the origin. Thus K_1 is both closed and bounded, and therefore compact.

2. (a) A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed. A topological space X is said to be *path-connected* if and only if, given any two points x_0 and x_1 of X, there exists a path in X from x_0 to x_1.

(b) Suppose that X is connected. Let $f: X \to \mathbb{Z}$ be a continuous function. Choose $n \in f(X)$, and let

$$U = \{x \in X : f(x) = n\}, \qquad V = \{x \in X : f(x) \neq n\}.$$

Then U and V are the preimages of the open subsets $\{n\}$ and $\mathbb{Z} \setminus \{n\}$ of $\mathbb{Z}$, and therefore both U and V are open in X. Moreover $U \cap V = \emptyset$, and $X = U \cup V$. It follows that $V = X \setminus U$, and thus U is both open and closed. Moreover U is non-empty, since $n \in f(X)$. It follows from the connectedness of X that $U = X$, so that $f: X \to \mathbb{Z}$ is constant, with value n.

Conversely suppose that every continuous function $f: X \to \mathbb{Z}$ is constant. Let S be a subset of X which is both open and closed. Let $f: X \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \notin S. \end{cases}$$

Now the preimage of any subset of $\mathbb{Z}$ under f is one of the open sets $\emptyset$, S, $X \setminus S$ and X. Therefore the function f is continuous. But then the function f is constant, so that either $S = \emptyset$ or $S = X$. This shows that X is connected.

(c) Let X be a path-connected topological space, and let $f: X \to \mathbb{Z}$ be a continuous integer-valued function on X. If x_0 and x_1 are any two points of X then there exists a path $\gamma: [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. But then $f \circ \gamma: [0, 1] \to \mathbb{Z}$ is a continuous integer-valued function on $[0, 1]$. But $[0, 1]$ is connected, therefore $f \circ \gamma$ is constant. It follows that $f(x_0) = f(x_1)$. Thus every continuous integer-valued function on X is constant. Therefore X is connected.

(d) Suppose $f(X) \neq \mathbb{R}$. Then there would exist a real number t such that $t \in \mathbb{R} \setminus f(X)$. Let

$$U = \{x \in X : f(x) < t\}, \quad V = \{x \in X : f(x) < t\}.$$

Then U and V would be non-empty open sets, $U \cap V = \emptyset$ and $X = U \cup V$. But then $U = X \setminus V$, and therefore U would be a subset of X that was both open and closed, but was neither $\emptyset$ nor X. But this would contradict the connectedness of X. Therefore $f(X) = \mathbb{R}$.

3. (a) Let X and $\tilde{X}$ be topological spaces and let $p: \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p: \tilde{X} \to X$ is said to be a *covering map* if $p: \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) Given a point (x_0, y_0) of $\mathbb{R}^2$ Let U be an open set in $\mathbb{R}^2$. Then

$$q^{-1}(q(U)) = \bigcup_{(j,k)\in\mathbb{Z}^2} (U + (j,k)),$$

where $U + (j,k) = \{(x + j, y + k) : (x,y) \in U\}$. It follows that, given any open set U in $\mathbb{R}^2$, $q^{-1}(q(U))$ is a union of open sets, and is thus itself an open set in T^2.

Suppose that the open set U is contained within an open square of the form

$$\{(x,y) \in \mathbb{R}^2 : x_0 - \frac{1}{2} < x < x_0 + \frac{1}{2} \text{ and } y_0 - \frac{1}{2} < y < y_0 + \frac{1}{2}\}.$$

Then $U \cap U + (j,k) = \emptyset$ whenever $j \neq 0$ or $k \neq 0$. It follows that $q|U$ maps U bijectively onto $q(U)$. Moreover it maps open sets to open sets, and therefore the inverse of this bijection is continuous. It follows that $q|U$ maps U homeomorphically onto $q(U)$. We see therefore that $q^{-1}(q(U))$ is the disjoint union of the open sets $U + (j,k)$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$, and each of these open sets is mapped homeomorphically onto $q(U)$ by q. Thus $q(U)$ is evenly covered by the map $q: \mathbb{R}^2 \to T^2$, provided that U is contained within an open square of the form given above. The map $q: \mathbb{R}^2 \to T^2$ is surjective, and each point of T^2 is contained within an evenly covered open set of the type just described. Therefore $q: \mathbb{R}^2 \to T^2$ is a covering map.

(c) Consider the map $g: \mathbb{R}^2 \to T^2$ that sends a point (x,y) of $\mathbb{R}^2$ to the point $q(nx, y)$. If (x_1, y_1) and (x_2, y_2) are points of $\mathbb{R}^2$, and if $q(x_1, y_1) = q(x_2, y_2)$, then $x_1 - x_2$ and $y_1 - y_2$ are integers. But then $q(nx_1, y_1) = q(nx_2, y_2)$, since $nx_1 - nx_2$ and $y_1 - y_2$ are both integers. It follows that there exists a well-defined function $f_n: T^2 \to T^2$ such that $f_n(q(x,y)) = q(nx, y)$ for all points (x,y) of $\mathbb{R}^2$. Moreover $f_n \circ q: \mathbb{R}^2 \to T^2$ is continuous, and $q: \mathbb{R}^2 \to T^2$ is an identification map. It follows that $f_n: T^2 \to T^2$ is continuous, as required.

4. We regard S^1 as the unit circle in $\mathbb{R}^2$. Without loss of generality, we can take $b = (1, 0)$. Now the map $p\colon \mathbb{R} \to S^1$ which sends $t \in \mathbb{R}$ to $(\cos 2\pi t, \sin 2\pi t)$ is a covering map, and $b = p(0)$. Moreover $p(t_1) = p(t_2)$ if and only if $t_1 - t_2$ is an integer; in particular $p(t) = b$ if and only if t is an integer.

Let α and β be loops in S^1 based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be paths in $\mathbb{R}$ that satisfy $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that α and β represent the same element of $\pi_1(S^1, b)$. Then there exists a homotopy $F\colon [0,1] \times [0,1] \to S^1$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all $t \in [0,1]$, and $F(0, \tau) = F(1, \tau) = b$ for all $\tau \in [0,1]$. It follows from the Monodromy Theorem that this homotopy lifts to a continuous map $G\colon [0,1] \times [0,1] \to \mathbb{R}$ satisfying $p \circ G = F$. Moreover $G(0, \tau)$ and $G(1, \tau)$ are integers for all $\tau \in [0,1]$, since $p(G(0,\tau)) = b = p(G(1,\tau))$. Also $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$ are integers for all $t \in [0,1]$, since $p(G(t,0)) = \alpha(t) = p(\tilde{\alpha}(t))$ and $p(G(t,1)) = \beta(t) = p(\tilde{\beta}(t))$. Now any continuous integer-valued function on $[0,1]$ is constant, by the Intermediate Value Theorem. In particular the functions sending $\tau \in [0,1]$ to $G(0,\tau)$ and $G(1,\tau)$ are constant, as are the functions sending $t \in [0,1]$ to $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$. Thus

$$G(0,0) = G(0,1), \qquad G(1,0) = G(1,1),$$

$$G(1,0) - \tilde{\alpha}(1) = G(0,0) - \tilde{\alpha}(0), \qquad G(1,1) - \tilde{\beta}(1) = G(0,1) - \tilde{\beta}(0).$$

On combining these results, we see that

$$\tilde{\alpha}(1) - \tilde{\alpha}(0) = G(1,0) - G(0,0) = G(1,1) - G(0,1) = \tilde{\beta}(1) - \tilde{\beta}(0).$$

We conclude from this that there exists a well-defined function

$$\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$$

characterized by the property that $\lambda([\alpha]) = \tilde{\alpha}(1) - \tilde{\alpha}(0)$ for all loops α based at b, where $\tilde{\alpha}\colon [0,1] \to \mathbb{R}$ is any path in $\mathbb{R}$ satisfying $p \circ \tilde{\alpha} = \alpha$.

Next we show that λ is a homomorphism. Let α and β be any loops based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β. The element $[\alpha][\beta]$ of $\pi_1(S^1, b)$ is represented by the product path $\alpha.\beta$, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Define a continuous path $\sigma\colon [0,1] \to \mathbb{R}$ by

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \tilde{\beta}(2t-1) + \tilde{\alpha}(1) - \tilde{\beta}(0) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

(Note that $\sigma(t)$ is well-defined when $t = \frac{1}{2}$.) Then $p \circ \sigma = \alpha.\beta$ and thus

$$\begin{aligned}\lambda([\alpha][\beta]) &= \lambda([\alpha.\beta]) = \sigma(1) - \sigma(0) = \tilde{\alpha}(1) - \tilde{\alpha}(0) + \tilde{\beta}(1) - \tilde{\beta}(0) \\ &= \lambda([\alpha]) + \lambda([\beta]).\end{aligned}$$

Thus $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is a homomorphism.

Now suppose that $\lambda([\alpha]) = \lambda([\beta])$. Let $F\colon [0,1] \times [0,1] \to S^1$ be the homotopy between α and β defined by

$$F(t, \tau) = p\left((1-\tau)\tilde{\alpha}(t) + \tau\tilde{\beta}(t)\right),$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the lifts of α and β respectively starting at 0. Now $\tilde{\beta}(1) = \lambda([\beta]) = \lambda([\alpha]) = \tilde{\alpha}(1)$, and $\tilde{\beta}(0) = \tilde{\alpha}(0) = 0$. Therefore $F(0, \tau) = b = p(\tilde{\alpha}(1)) = F(1, \tau)$ for all $\tau \in [0, 1]$. Thus $\alpha \simeq \beta$ rel $\{0, 1\}$, and therefore $[\alpha] = [\beta]$. This shows that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is injective.

The homomorphism λ is surjective, since $n = \lambda([\gamma_n])$ for all $n \in \mathbb{Z}$, where the loop $\gamma_n\colon [0,1] \to S^1$ is given by

$$\gamma_n(t) = p(nt) = (\cos 2\pi nt, \sin 2\pi nt)$$

for all $t \in [0, 1]$. We conclude that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is an isomorphism.

5. (a) Let K be a simplicial complex which is a subdivision of some n-dimensional simplex Δ. We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n, with the following properties:—

   - for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,
   - if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

   (b) Sperner's Lemma: *Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.*

   Proof. Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let

$$N(i_0, i_1, \ldots, i_q)$$

denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$. Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2(N(0, 1, \ldots, n-1) - M),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required.

(c) Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s\colon K \to L$ is a simplicial approximation to $r\colon \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r\colon \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

6. (a) The homomorphism $\partial_q\colon C_q(K) \to C_{q-1}(K)$, is characterized by the property that

$$\partial_q \left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle\right) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned}
&\partial_{q-1}\partial_q \left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle\right) \\
&\qquad = \sum_{j=0}^{q} (-1)^j \partial_{q-1} \left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle\right) \\
&\qquad = \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&\qquad\quad + \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle \\
&\qquad = 0
\end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1} \circ \partial_q$ is determined by its values on all oriented q-simplices of K.

(b) Let K be a simplicial complex. A q-chain z is said to be a *q-cycle* if $\partial_q z = 0$. A q-chain b is said to be a *q-boundary* if $b = \partial_{q+1} c'$ for some $(q+1)$-chain c'. The group of q-cycles of K is denoted by $Z_q(K)$, and the group of q-boundaries of K is denoted by $B_q(K)$. Thus $Z_q(K)$ is the kernel of the boundary homomorphism $\partial_q\colon C_q(K) \to C_{q-1}(K)$, and $B_q(K)$ is the image of the boundary homomorphism $\partial_{q+1}\colon C_{q+1}(K) \to C_q(K)$. However $\partial_q \circ \partial_{q+1} = 0$, and therefore $B_q(K) \subset Z_q(K)$. We can therefore form the quotient group $H_q(K)$, where $H_q(K) = Z_q(K)/B_q(K)$. The group $H_q(K)$ is referred to as the *qth homology group* of the simplicial complex K.

(c) There is a well-defined homomorphism $D_q\colon C_q(K) \to C_{q+1}(K)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Now $\partial_1(D_0(\mathbf{v})) = \mathbf{v} - \mathbf{w}$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle - \left(\sum_{r=1}^{s} n_r \right) \langle \mathbf{w} \rangle = \sum_{r=1}^{s} n_r (\langle \mathbf{v}_r \rangle - \langle \mathbf{w} \rangle) \in B_0(K)$$

for all $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$. But $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition), and thus $H_0(K) = C_0(K)/B_0(K)$. It follows that there is a well-defined surjective homomorphism from $H_0(K)$ to $\mathbb{Z}$ induced by the homomorphism from $C_0(K)$ to $\mathbb{Z}$ that sends $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$ to $\sum_{r=1}^{s} n_r$. Moreover this induced homomorphism is an isomorphism from $H_0(K)$ to $\mathbb{Z}$.
Now let $q > 0$. Then

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)) \\
&= \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) \\
&= \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&= \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle))
\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K)$, and hence $Z_q(K) = B_q(K)$. It follows that $H_q(K)$ is the zero group for all $q > 0$, as required.

7. (a) A 1-chain c_1 of the simplicial complex is of the form

$$\begin{aligned} c_1 &= n_1\langle \mathbf{a}, \mathbf{b}\rangle + n_2\langle \mathbf{b}, \mathbf{c}\rangle + n_3\langle \mathbf{c}, \mathbf{d}\rangle + n_4\langle \mathbf{d}, \mathbf{a}\rangle + n_5\langle \mathbf{a}, \mathbf{c}\rangle \\ &\quad + n_6\langle \mathbf{a}, \mathbf{e}\rangle + n_7\langle \mathbf{b}, \mathbf{e}\rangle + n_8\langle \mathbf{c}, \mathbf{e}\rangle + n_9\langle \mathbf{d}, \mathbf{e}\rangle \end{aligned}$$

We see that

$$\begin{aligned} \partial_1 c_1 &= (n_4 - n_1 - n_5 - n_6)\langle \mathbf{a}\rangle + (n_1 - n_2 - n_7)\langle \mathbf{b}\rangle \\ &\quad + (n_2 - n_3 + n_5 - n_8)\langle \mathbf{c}\rangle + (n_3 - n_4 - n_9)\langle \mathbf{d}\rangle \\ &\quad + (n_6 + n_7 + n_8 + n_9)\langle \mathbf{e}\rangle \end{aligned}$$

It follows that $\partial_1 c_1 = 0$ if and only if

$$\begin{aligned} n_4 - n_1 - n_5 - n_6 &= 0, \\ n_1 - n_2 - n_7 &= 0, \\ n_2 - n_3 + n_5 - n_8 &= 0, \\ n_3 - n_4 - n_9 &= 0 \end{aligned}$$

We solve for n_5, n_6, n_7, n_8 and n_9 in terms of n_1, n_2, n_3, n_4 and n_5. Now $n_6 = n_4 - n_1 - n_5$, $n_7 = n_1 - n_2$, $n_8 = n_2 - n_3 + n_5$, $n_9 = n_3 - n_4$. Now, these equations for $n_6, \ldots, n_9$ are sufficient to ensure that $n_6 + n_7 + n_8 + n_9 = 0$. Therefore $\partial_1 c_1 = 0$ if and only if

$$\begin{aligned} c_1 &= n_1\langle \mathbf{a}, \mathbf{b}\rangle + n_2\langle \mathbf{b}, \mathbf{c}\rangle + n_3\langle \mathbf{c}, \mathbf{d}\rangle + n_4\langle \mathbf{d}, \mathbf{a}\rangle + n_5\langle \mathbf{a}, \mathbf{c}\rangle \\ &\quad + (n_4 - n_1 - n_5)\langle \mathbf{a}, \mathbf{e}\rangle + (n_1 - n_2)\langle \mathbf{b}, \mathbf{e}\rangle \\ &\quad + (n_2 - n_3 + n_5)\langle \mathbf{c}, \mathbf{e}\rangle + (n_3 - n_4)\langle \mathbf{d}, \mathbf{e}\rangle \\ &= n_1 z_1 + n_2 z_2 + n_3 z_3 + n_4 z_4 + n_5 z_5 \end{aligned}$$

(b) A 2-chain c_2 is of the form

$$c_2 = k_1\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle + k_2\langle \mathbf{a}, \mathbf{d}, \mathbf{e}\rangle,$$

and

$$\begin{aligned} \partial_2 c_2 &= k_1\langle \mathbf{a}, \mathbf{b}\rangle + k_1\langle \mathbf{b}, \mathbf{c}\rangle + k_2\langle \mathbf{c}, \mathbf{d}\rangle + k_2\langle \mathbf{d}, \mathbf{a}\rangle \\ &\quad + (k_2 - k_1)\langle \mathbf{a}, \mathbf{c}\rangle \\ &= k_1 z_1 + k_1 z_2 + k_2 z_3 + k_2 z_4 + (k_2 - k_1) z_5 \end{aligned}$$

Thus $n_1 z_1 + n_2 z_2 + n_3 z_3 + n_4 z_4 + n_5 z_5$ is a 1-boundary if and only if $n_2 = n_1$, $n_4 = n_3$ and $n_5 = n_3 - n_1$.

(c) Consider the homomorphism

$$\theta\colon Z_1(K) \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

defined by

$$\theta(n_1 z_1 + n_2 z_2 + n_3 z_3 + n_4 z_4 + n_5 z_5) = (n_2 - n_1, n_4 - n_3, n_5 - n_3 + n_1).$$

This homomorphism is obviously surjective, and it follows from (b) that the kernel of the homomorphism is $B_1(K)$. Therefore

$$H_1(K) = Z_1(K)/\ker\theta \cong \theta(Z_1(K)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

as required.

8. (a) A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of Abelian groups, together with homomorphisms $\partial_i: C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i: C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1}: C_{i+1} \to C_i$.

Let C_* and D_* be chain complexes. A *chain map* $f: C_* \to D_*$ is a sequence $f_i: C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*: A_* \to B_*$ and $q_*: B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

(b) Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i: B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2}: A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_i$, by exactness. But then

$$\begin{aligned} p_{i-1}(w + \partial_i(a)) &= p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) \\ &= \partial_i(b') = p_{i-1}(w'), \end{aligned}$$

and $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i \colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function $\tilde{\alpha}_i$ is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1} \colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i \colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism

$$\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*),$$

as required.

9. (a) Let K be a simplicial complex and let L and M be subcomplexes of K such that $K = L \cup M$. Let

$$i_q\colon C_q(L \cap M) \to C_q(L), \qquad j_q\colon C_q(L \cap M) \to C_q(M),$$

$$u_q\colon C_q(L) \to C_q(K), \qquad v_q\colon C_q(M) \to C_q(K)$$

be the inclusion homomorphisms induced by the inclusion maps $i\colon L \cap M \hookrightarrow L$, $j\colon L \cap M \hookrightarrow M$, $u\colon L \hookrightarrow K$ and $v\colon M \hookrightarrow K$. Then

$$0 \longrightarrow C_*(L \cap M) \xrightarrow{k_*} C_*(L) \oplus C_*(M) \xrightarrow{w_*} C_*(K) \longrightarrow 0$$

is a short exact sequence of chain complexes, where

$$\begin{aligned} k_q(c) &= (i_q(c), -j_q(c)), \\ w_q(c', c'') &= u_q(c') + v_q(c''), \\ \partial_q(c', c'') &= (\partial_q(c'), \partial_q(c'')) \end{aligned}$$

for all $c \in C_q(L \cap M)$, $c' \in C_q(L)$ and $c'' \in C_q(M)$, and this gives rise to a long exact sequence

$$\cdots \xrightarrow{\alpha_{q+1}} H_q(L \cap M) \xrightarrow{k_*} H_q(L) \oplus H_q(M) \xrightarrow{w_*} H_q(K) \xrightarrow{\alpha_q} H_{q-1}(L \cap M) \xrightarrow{k_*} \cdots,$$

of homology groups. This long exact sequence of homology groups is referred to as the *Mayer-Vietoris sequence* associated with the decomposition of K as the union of the subcomplexes L and M.

(b) Let $\mathbf{v}$ be a vertex of $L \cap M$. Then the homology class of $\mathbf{v}$ in the respective groups generates $H_0(L \cap M)$, $H_0(L)$ and $H_0(M)$. It follows that $i_*\colon H_0(L \cap M) \to H_0(L)$ and $j_*\colon H_0(L \cap M) \to H_0(M)$ are isomorphisms, and therefore $k_*\colon H_0(L \cap M) \to H_0(L) \oplus H_0(M)$ is a monomorphism. It follows from the exactness of the Mayer-Vietoris seqence that the homomorphism $H_1(K) \to H_0(L \cap M)$ in that sequence is the zero homomorphism, from which it follows (by exactness) that $w_*\colon H_1(L) \oplus H_1(M) \to H_1(K)$ is exact. Now $H_1(L) \oplus H_1(M) \cong H_1(L)$, since $H_1(M) = 0$, and the homomorphisms $k_*\colon H_1(L \cap M) \to H_1(L) \oplus H_1(M)$ and $w_*\colon H_1(L) \oplus H_1(M) \to H_1(K)$ correspond to the homomorphism i_* and u_* respectively. Also $H_2(L) = 0$ and $H_2(M) = 0$. Therefore the Mayer-Vietoris sequence yields the following exact sequence:

$$0 \longrightarrow H_2(K) \longrightarrow H_1(L \cap M) \xrightarrow{i_*} H_1(L) \xrightarrow{u_*} H_1(K) \longrightarrow 0.$$

The fact that $i_*([z_0]) = 2[z_1]$ where $[z_0]$ and $[z_1]$ generate $H_1(L \cap M)$ and $H_1(L)$ respectively, ensures that $H_1(L)/i_*(H_1(L \cap M)) \cong \mathbb{Z}/2\mathbb{Z}$. It follows that

$$H_1(K) \cong \frac{H_1(L)}{i_*(H_1(L \cap M))} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ denotes the cyclic group of order 2. Also $H_2(K) \cong ker(i_*\colon H_1(L \cap M) \to H_1(L))$, and therefore $H_2(K) = 0$. The fact that K is connected ensures that $H_0(K) = 0$. This also follows from the fact that

$$H_0(K) \cong \frac{H_0(L) \oplus H_0(M)}{i_*(H_0(L \cap M))} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\{(n, -n) : n \in \mathbb{Z}\}} \cong \mathbb{Z}.$$