Course 421 - Trinity Term 2003: Worked Solutions

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Many of the questions represent bookwork (and the worked solutions presented here tend to follow the printed lecture notes). The exceptions are the following questions or parts of questions:—

Question 1 (b) Not bookwork.

Question 1 (d) Not stated in lecture notes in this particular form, but this, or related, results form part of the proof of theorems like the Path Lifting Theorem.

Question 2 (b) Not bookwork. There was a question on a 421 examination paper some years ago, which was equivalent to asking candidates to show that the function ν is constant, provided that X is connected.

Question 2 (c) Not bookwork. There are questions like this on previous examination papers, and in problem sets.

Question 6 This is not bookwork. However similar (but not identical) questions have regularly appeared on previous 421 papers which I have set; and I have gone through a similar problem (involving the homology groups of the octohedron) in class.

Question 8 (b) Not bookwork. An exercise on finding the homology groups of the real projective plane using the Mayer-Vietoris sequence is to be found on a problem set, but not phrased as in this examination question.

Each question will be marked out of 20. The final result may well be the sum of the results of the best six questions, converted to a percentage; but a piecewise linear re-scaling may be applied if this is felt to be appropriate. The parameters of such a rescaling would be determined after the individual questions on the scripts have been marked.

1. (a) [Quoted from printed lecture notes.] Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

   If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

   A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

   (b) Let $\mathcal{V}$ be an open cover of $\mathbb{R}$ with the Zariski topology. Choose $x_0 \in \mathbb{R}$. Then $x_0 \in V_0$ for some open set V_0 belonging to the open cover $\mathcal{V}$. Now V is non-empty. It follows from the definition of the Zariski topology that $\mathbb{R} \setminus V_0$ is a finite set. Let $\mathbb{R} \setminus V_0 = \{x_1, x_2, \ldots, x_k\}$. Now there exist open sets $V_1, \ldots, V_k$ belonging to the open cover $\mathcal{V}$ such that $x_i \in V_i$ for $i = 1, 2, \ldots, k$. Then the open sets $V_0, V_1, \ldots, V_k$ constitute a finite subcover of the open cover $\mathcal{V}$.

   (c) [Quoted from printed lecture notes.] **(Lebesgue Lemma)** *Let (X, d) be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.*

   Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

   where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let $\delta > 0$ be given by

$$\delta = \text{minimum}(\delta_1, \delta_2, \ldots, \delta_r).$$

   Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i

between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{U}$, as required.

(d) It follows from the continuity of $f\colon [0,1] \to X$ that the collection $\mathcal{W}$ of subsets of $[0,1]$ that are of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$ is an open cover of $[0,1]$. The topological space $[0,1]$ is a compact metric space. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that every subset of $[0,1]$ whose diameter is less than δ is contained in some set belonging to the open cover $\mathcal{W}$ of $[0,1]$, and is therefore contained in $f^{-1}(V)$ for some open set V belonging to $\mathcal{V}$. We may then choose $t_0, t_1, \ldots, t_m$ such that $t_i - t_{i-1} < \delta$ for $i = 1, 2, \ldots, m$.

2. (a) [Quoted from lecture notes.] Let X and $\tilde{X}$ be topological spaces and let $p\colon \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p\colon \tilde{X} \to X$ is said to be a *covering map* if $p\colon \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) Let U be an evenly covered open set in X. Then $p^{-1}(U)$ is a disjoint union of open sets $V_1, \ldots, V_k$, where V_i is mapped homeomorphically onto U by p for $i = 1, 2, \ldots, k$. If $u \in U$ then $p^{-1}(\{u\})$ intersects V_i in a single point for $i = 1, 2, \ldots, k$, and therefore $\nu(u) = k$ for all $u \in U$. It follows that the function $\nu\colon X \to \mathbb{Z}$ is constant on each evenly covered open set.

Now each point of X belongs to some evenly covered open set. Therefore, given any point in X, the function ν is constant over a neighbourhood of that point. Such a function is continuous. [For each $k \in \mathbb{Z}$, $\nu^{-1}(k)$ can be expressed as a union of (evenly covered) open sets, and is therefore itself an open sets. It follows that $\nu^{-1}(W)$ is an open set in X for any subset W of $\mathbb{Z}$, since it is a union of open sets of the form $\nu^{-1}(k)$.]

(c) (i) The map $f\colon A_1 \to A_2$ is a covering map. Indeed let W_1 and W_2 be the open subsets of A_2 defined by

$$W_1 = A_2 \setminus P, \quad W_2 = A_2 \setminus (-P),$$

where P is the set of positive real numbers, and $-P$ is the set of negative real numbers. Then Then $f^{-1}(W_1) = V_{1+} \cup V_{1-}$, and $f^{-1}(W_2) = V_{2+} \cup V_{2-}$, where $V_{1\pm}$ and $V_{2\pm}$ are the open subsets of A_1 defined by

$$V_{1+} = \{z \in A_1 : \operatorname{Im} z > 0\}, \quad V_{1-} = \{z \in A_1 : \operatorname{Im} z < 0\},$$

$$V_{2+} = \{z \in A_1 : \operatorname{Re} z > 0\}, \quad V_{2-} = \{z \in A_1 : \operatorname{Re} z < 0\}.$$

Moreover V_{1+} and V_{1-} are each mapped homeomorphically onto W_1 under f, and similarly V_{2+} and V_{2-} are each mapped homeomorphically onto W_2 under f. Therefore W_1 and W_2 are both evenly covered by f. Moreover $f\colon A_1 \to A_2$ is surjective, and every point of A_2 belongs to one or other of the evenly covered open sets W_1 and W_2. Thus $f\colon A_1 \to A_2$ is a covering map.

(ii) The map $g\colon A_1 \setminus P \to A_2$ is not a covering map. Indeed, if $\nu\colon A_2 \to \mathbb{Z}$ is defined as described in (b) then $\nu(z) = 1$ if $z \in A_2 \cap P$, whereas $\nu(2) = 2$ if $z \in A_2 \setminus P$, where P is the set of positive real numbers. Therefore the function ν is not continuous on A_2, which it would have to be were the map g a covering map.

3. [Quoted from lecture notes.] We regard S^1 as the unit circle in $\mathbb{R}^2$. Without loss of generality, we can take $b = (1, 0)$. Now the map $p\colon \mathbb{R} \to S^1$ which sends $t \in \mathbb{R}$ to $(\cos 2\pi t, \sin 2\pi t)$ is a covering map, and $b = p(0)$. Moreover $p(t_1) = p(t_2)$ if and only if $t_1 - t_2$ is an integer; in particular $p(t) = b$ if and only if t is an integer.

   Let α and β be loops in S^1 based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be paths in $\mathbb{R}$ that satisfy $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that α and β represent the same element of $\pi_1(S^1, b)$. Then there exists a homotopy $F\colon [0, 1] \times [0, 1] \to S^1$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all $t \in [0, 1]$, and $F(0, \tau) = F(1, \tau) = b$ for all $\tau \in [0, 1]$. It follows from the Monodromy Theorem that this homotopy lifts to a continuous map $G\colon [0, 1] \times [0, 1] \to \mathbb{R}$ satisfying $p \circ G = F$. Moreover $G(0, \tau)$ and $G(1, \tau)$ are integers for all $\tau \in [0, 1]$, since $p(G(0, \tau)) = b = p(G(1, \tau))$. Also $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$ are integers for all $t \in [0, 1]$, since $p(G(t, 0)) = \alpha(t) = p(\tilde{\alpha}(t))$ and $p(G(t, 1)) = \beta(t) = p(\tilde{\beta}(t))$. Now any continuous integer-valued function on $[0, 1]$ is constant, by the Intermediate Value Theorem. In particular the functions sending $\tau \in [0, 1]$ to $G(0, \tau)$ and $G(1, \tau)$ are constant, as are the functions sending $t \in [0, 1]$ to $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$. Thus

   $$G(0, 0) = G(0, 1), \qquad G(1, 0) = G(1, 1),$$

   $$G(1, 0) - \tilde{\alpha}(1) = G(0, 0) - \tilde{\alpha}(0), \qquad G(1, 1) - \tilde{\beta}(1) = G(0, 1) - \tilde{\beta}(0).$$

   On combining these results, we see that

   $$\tilde{\alpha}(1) - \tilde{\alpha}(0) = G(1, 0) - G(0, 0) = G(1, 1) - G(0, 1) = \tilde{\beta}(1) - \tilde{\beta}(0).$$

   We conclude from this that there exists a well-defined function $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ characterized by the property that $\lambda([\alpha]) = \tilde{\alpha}(1) - \tilde{\alpha}(0)$ for all loops α based at b, where $\tilde{\alpha}\colon [0, 1] \to \mathbb{R}$ is any path in $\mathbb{R}$ satisfying $p \circ \tilde{\alpha} = \alpha$.

   Next we show that λ is a homomorphism. Let α and β be any loops based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β. The element $[\alpha][\beta]$ of $\pi_1(S^1, b)$ is represented by the product path $\alpha.\beta$, where

   $$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t - 1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

   Define a continuous path $\sigma\colon [0, 1] \to \mathbb{R}$ by

   $$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \tilde{\beta}(2t - 1) + \tilde{\alpha}(1) - \tilde{\beta}(0) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

(Note that $\sigma(t)$ is well-defined when $t = \frac{1}{2}$.) Then $p \circ \sigma = \alpha.\beta$ and thus

$$\begin{aligned}\lambda([\alpha][\beta]) &= \lambda([\alpha.\beta]) = \sigma(1) - \sigma(0) = \tilde{\alpha}(1) - \tilde{\alpha}(0) + \tilde{\beta}(1) - \tilde{\beta}(0) \\ &= \lambda([\alpha]) + \lambda([\beta]).\end{aligned}$$

Thus $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is a homomorphism.

Now suppose that $\lambda([\alpha]) = \lambda([\beta])$. Let $F\colon [0,1] \times [0,1] \to S^1$ be the homotopy between α and β defined by

$$F(t,\tau) = p\left((1-\tau)\tilde{\alpha}(t) + \tau\tilde{\beta}(t)\right),$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the lifts of α and β respectively starting at 0. Now $\tilde{\beta}(1) = \lambda([\beta]) = \lambda([\alpha]) = \tilde{\alpha}(1)$, and $\tilde{\beta}(0) = \tilde{\alpha}(0) = 0$. Therefore $F(0,\tau) = b = p(\tilde{\alpha}(1)) = F(1,\tau)$ for all $\tau \in [0,1]$. Thus $\alpha \simeq \beta$ rel $\{0,1\}$, and therefore $[\alpha] = [\beta]$. This shows that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is injective.

The homomorphism λ is surjective, since $n = \lambda([\gamma_n])$ for all $n \in \mathbb{Z}$, where the loop $\gamma_n\colon [0,1] \to S^1$ is given by

$$\gamma_n(t) = p(nt) = (\cos 2\pi nt, \sin 2\pi nt)$$

for all $t \in [0,1]$. We conclude that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is an isomorphism.

4. (a) A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$. A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The *polyhedron* of a simplicial complex K is the union of all the simplices of K.

(b) Let K be a simplicial complex, and let $\mathbf{x} \in |K|$. The *star* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in K is the union of the interiors of all simplices of K that contain the point $\mathbf{x}$.

(c) Let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of simplicial complexes K and L. A simplicial map $s\colon K \to L$ is said to be a *simplicial approximation* to f if, for each $\mathbf{x} \in |K|$, $s(\mathbf{x})$ is an element of the unique simplex of L which contains $f(\mathbf{x})$ in its interior.

(d) Let $s\colon K \to L$ be a simplicial approximation to $f\colon |K| \to |L|$, let $\mathbf{v}$ be a vertex of K, and let $\mathbf{x} \in \mathrm{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of σ, by definition of $\mathrm{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to τ (since s is a simplicial approximation to the map f), and therefore $s(\mathbf{x})$ must belong to the interior of some face of τ. But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, since $\mathbf{x}$ is in the interior of σ. It follows that $s(\sigma)$ must be a face of τ, and therefore $s(\mathbf{v})$ must be a vertex of τ. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$, then $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$.

Conversely let $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ be a function with the property that $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$ for all vertices $\mathbf{v}$ of K. Let $\mathbf{x}$ be a point in the interior of some simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L\left(s(\mathbf{v}_j)\right)$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of τ, and $s(\mathbf{x}) \in \tau$. We conclude that the function $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ represents a simplicial map which is a simplicial approximation to $f\colon |K| \to |L|$, as required.

(e) The collection consisting of the stars $\mathrm{st}_L(\mathbf{w})$ of all vertices $\mathbf{w}$ of L is an open cover of $|L|$, since each star $\mathrm{st}_L(\mathbf{w})$ is open in $|L|$ and the interior of any simplex of L is contained in $\mathrm{st}_L(\mathbf{w})$ whenever $\mathbf{w}$ is a vertex of that simplex. It follows from the continuity of the map $f\colon |K| \to |L|$ that the collection consisting of the preimages $f^{-1}(\mathrm{st}_L(\mathbf{w}))$ of the stars of all vertices $\mathbf{w}$ of L is an open cover of $|K|$. It then follows from the Lebesgue Lemma that there exists some $\delta > 0$ with the property that every subset of $|K|$ whose diameter is less than δ is mapped by f into $\mathrm{st}_L(\mathbf{w})$ for some vertex $\mathbf{w}$ of L.

Now the mesh $\mu(K^{(j)})$ of the jth barycentric subdivision of K tends to zero as $j \to +\infty$, since

$$\mu(K^{(j)}) \leq \left(\frac{\dim K}{\dim K + 1}\right)^j \mu(K)$$

for all j. Thus we can choose j such that $\mu(K^{(j)}) < \frac{1}{2}\delta$. If $\mathbf{v}$ is a vertex of $K^{(j)}$ then each point of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is within a distance $\frac{1}{2}\delta$ of $\mathbf{v}$, and hence the diameter of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is at most δ. We can therefore choose, for each vertex $\mathbf{v}$ of $K^{(j)}$ a vertex $s(\mathbf{v})$ of L such that $f(\mathrm{st}_{K^{(j)}}(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$. In this way we obtain a function $s\colon \mathrm{Vert}\, K^{(j)} \to \mathrm{Vert}\, L$ from the vertices of $K^{(j)}$ to the vertices of L. This is the desired simplicial approximation to f.

5. (a) [Quoted from lecture notes.] Let K be a simplicial complex, and let $\mathbf{y}$ and $\mathbf{z}$ be vertices of K. We say that $\mathbf{y}$ and $\mathbf{z}$ can be joined by an *edge path* if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of K with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to K for $j = 1, 2, \ldots, m$.

(b) [Quoted from lecture notes.] It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required.

(c) [Quoted from lecture notes.] Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be the vertices of the simplicial complex K. Every 0-chain of K can be expressed uniquely as a formal sum of the form

$$n_1\langle \mathbf{u}_1 \rangle + n_2\langle \mathbf{u}_2 \rangle + \cdots + n_r\langle \mathbf{u}_r \rangle$$

for some integers $n_1, n_2, \ldots, n_r$. It follows that there is a well-defined homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ defined by

$$\varepsilon\left(n_1\langle \mathbf{u}_1 \rangle + n_2\langle \mathbf{u}_2 \rangle + \cdots + n_r\langle \mathbf{u}_r \rangle\right) = n_1 + n_2 + \cdots + n_r.$$

Now $\varepsilon(\partial_1(\langle \mathbf{y}, \mathbf{z} \rangle)) = \varepsilon(\langle \mathbf{z} \rangle - \langle \mathbf{y} \rangle) = 0$ whenever $\mathbf{y}$ and $\mathbf{z}$ are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0$, and hence $B_0(K) \subset \ker \varepsilon$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ be vertices of K determining an edge path. Then

$$\langle \mathbf{v}_m \rangle - \langle \mathbf{v}_0 \rangle = \partial_1 \left(\sum_{j=1}^{m} \langle \mathbf{v}_{j-1}, \mathbf{v}_j \rangle \right) \in B_0(K).$$

Now $|K|$ is connected, and therefore any pair of vertices of K can be joined by an edge path. We deduce that $\langle \mathbf{z} \rangle - \langle \mathbf{y} \rangle \in B_0(K)$ for all vertices $\mathbf{y}$ and $\mathbf{z}$ of K. Thus if $c \in \ker \varepsilon$, where $c = \sum_{j=1}^{r} n_j \langle \mathbf{u}_j \rangle$, then $\sum_{j=1}^{r} n_j = 0$, and hence $c = \sum_{j=2}^{r} n_j (\langle \mathbf{u}_j \rangle - \langle \mathbf{u}_1 \rangle)$. But $\langle \mathbf{u}_j \rangle - \langle \mathbf{u}_1 \rangle \in B_0(K)$. It follows that $c \in B_0(K)$. We conclude that $\ker \varepsilon \subset B_0(K)$, and hence $\ker \varepsilon = B_0(K)$.

Now the homomorphism $\varepsilon \colon C_0(K) \to \mathbb{Z}$ is surjective and its kernel is $B_0(K)$. Therefore it induces an isomorphism from $C_0(K)/B_0(K)$ to $\mathbb{Z}$. However $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition). Thus $H_0(K) \equiv C_0(K)/B_0(K) \cong \mathbb{Z}$, as required.

6. (a)

$$\begin{aligned}\partial_2 c &= (n_{124}+n_{125})e_{12} + (n_{134}+n_{135})e_{13}\\ &\quad + (-n_{124}-n_{134})e_{14} + (-n_{125}-n_{135})e_{15}\\ &\quad + (n_{234}+n_{235})e_{23} + (n_{124}-n_{234})e_{24}\\ &\quad + (n_{125}-n_{235})e_{25} + (n_{134}+n_{234})e_{34}\\ &\quad + (n_{135}+n_{235})e_{35}.\end{aligned}$$

It follows that $\partial_2 c = 0$ if and only if

$$n_{124} = -n_{125} = -n_{134} = n_{135} = n_{234} = -n_{235}.$$

Morover $B_2(K) = 0$ (since $C_3(K) = 0$). It follows that

$$H_2(K) = Z_2(K)/B_2(K) = Z_2(K) = \{nz : n \in \mathbb{Z}\} \cong \mathbb{Z},$$

where

$$z = \sigma_{124} - \sigma_{125} - \sigma_{134} + \sigma_{135} + \sigma_{234} - \sigma_{235}.$$

(b) Let

$$\begin{aligned}g &= m_{12}e_{12} + m_{13}e_{13} + m_{14}e_{14} + m_{15}e_{15} + m_{23}e_{23}\\ &\quad + m_{24}e_{24} + m_{25}e_{25} + m_{34}e_{34} + m_{35}e_{35}.\end{aligned}$$

Then

$$\begin{aligned}\partial_1 g &= (-m_{12}-m_{13}-m_{14}-m_{15})P_1\\ &\quad + (m_{12}-m_{23}-m_{24}-m_{25})P_2\\ &\quad + (m_{13}+m_{23}-m_{34}-m_{35})P_3\\ &\quad + (m_{14}+m_{24}+m_{34})P_4 + (m_{15}+m_{25}+m_{35})P_5\end{aligned}$$

If g is a 1-boundary of K, then g is a 1-cycle of K, since $\partial_1 \circ \partial_2 = 0$. [Alternatively, this may be verified directly.] Conversely suppose that g is a 1-cycle of K. Then $\partial_1 g = 0$, and thus

$$\begin{aligned}m_{12}+m_{13}+m_{14}+m_{15} &= 0,\\ m_{12}-m_{23}-m_{24}-m_{25} &= 0,\\ m_{13}+m_{23}-m_{34}-m_{35} &= 0,\\ m_{14}+m_{24}+m_{34} &= 0,\\ m_{15}+m_{25}+m_{35} &= 0.\end{aligned}$$

Suppose there exists a 2-chain c such that $\partial_2 c = g$, where c is as in part (a). We need to show that integers n_{124}, n_{125}, n_{134}, n_{135}, n_{234} and n_{235} may be found such that

$$\begin{aligned}
n_{124} + n_{125} &= m_{12}, \\
n_{134} + n_{135} &= m_{13}, \\
-n_{124} - n_{134} &= m_{14}, \\
-n_{125} - n_{135} &= m_{15}, \\
n_{234} + n_{235} &= m_{23}, \\
n_{124} - n_{234} &= m_{24}, \\
n_{125} - n_{235} &= m_{25}, \\
n_{134} + n_{234} &= m_{34}, \\
n_{135} + n_{235} &= m_{35}.
\end{aligned}$$

We look for a solution satisfying $n_{235} = 0$. [This is sensible since any 2-chain of K is homologous to a 2-chain satisfying this condition.] Take

$$\begin{aligned}
n_{235} &= 0, \\
n_{135} &= m_{35}, \\
n_{234} &= m_{23}, \\
n_{125} &= m_{25}, \\
n_{134} &= m_{13} - n_{135} = m_{13} - m_{35}, \\
n_{124} &= m_{24} + n_{234} = m_{23} + m_{24}.
\end{aligned}$$

One can verify directly that this solves the relevant equations. For example

$$n_{124} + n_{125} = m_{23} + m_{24} + m_{25} = m_{12}.$$

Thus any 1-cycle of K is a 1-boundary of K. We conclude that $Z_1(K) = B_1(K)$, and thus $H_1(K) = 0$.

(c) The calculation in (b) shows that any 1-cycle g of K is of the form $\partial_2 c$ for some 2-chain c of K with $n_{235} = 0$. Such a 2-chain c is in fact a 2-chain of L. It follows directly that $Z_1(L) = B_1(L)$, and thus $H_1(L) = 0$. It also follows directly from the answer to (a) that if c is a 2-cycle of L (i.e., if $\partial_2 c = 0$ and $n_{235} = 0$) then $c = 0$. Therefore $Z_2(L) = 0$, and hence $H_2(L) = 0$. [There are alternative ways of answering this part of the question.]

7. (a) [All definitions taken from from printed lecture notes.] The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of Abelian groups and homomorphisms is said to be *exact* at G if and only if $\mathrm{image}(p\colon F \to G) = \ker(q\colon G \to H)$. A sequence of Abelian groups and homomorphisms is said to be *exact* if it is exact at each Abelian group occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of Abelian groups, together with homomorphisms $\partial_i\colon C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i\colon C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1}\colon C_{i+1} \to C_i$.

Let C_* and D_* be chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence $f_i\colon C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

A *short exact sequence* $0 \to A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \to 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*\colon A_* \to B_*$ and $q_*\colon B_* \to C_*$ such that the sequence

$$0 \to A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \to 0$$

is exact for each integer i.

(b) [Quoted from from printed lecture notes.] Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i\colon B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \to A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2}\colon A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then

$q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_i$, by exactness. But then

$$\begin{aligned} p_{i-1}(w + \partial_i(a)) &= = p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) \\ &= \partial_i(b') = p_{i-1}(w'), \end{aligned}$$

and $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i: Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function $\tilde{\alpha}_i$ is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}: B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i: Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism

$$\alpha_i: H_i(C_*) \to H_{i-1}(A_*),$$

as required.

8. (a) [Adapted from printed lecture notes.] Let K be a simplicial complex and let L and M be subcomplexes of K such that $K = L \cup M$. Let

$$i_q\colon C_q(L \cap M) \to C_q(L), \qquad j_q\colon C_q(L \cap M) \to C_q(M),$$

$$u_q\colon C_q(L) \to C_q(K), \qquad v_q\colon C_q(M) \to C_q(K)$$

be the inclusion homomorphisms induced by the inclusion maps $i\colon L \cap M \hookrightarrow L$, $j\colon L \cap M \hookrightarrow M$, $u\colon L \hookrightarrow K$ and $v\colon M \hookrightarrow K$. Then

$$0 \to C_*(L \cap M) \xrightarrow{k_*} C_*(L) \oplus C_*(M) \xrightarrow{w_*} C_*(K) \to 0$$

is a short exact sequence of chain complexes, where

$$\begin{aligned} k_q(c) &= (i_q(c), -j_q(c)), \\ w_q(c', c'') &= u_q(c') + v_q(c''), \\ \partial_q(c', c'') &= (\partial_q(c'), \partial_q(c'')) \end{aligned}$$

for all $c \in C_q(L \cap M)$, $c' \in C_q(L)$ and $c'' \in C_q(M)$. This gives rise to an exact sequence

$$\cdots \xrightarrow{\alpha_{q+1}} H_q(L \cap M) \xrightarrow{k_*} H_q(L) \oplus H_q(M) \xrightarrow{w_*} H_q(K) \xrightarrow{\alpha_q} H_{q-1}(L \cap M) \xrightarrow{k_*} \cdots,$$

of homology groups. This long exact sequence of homology groups is referred to as the *Mayer-Vietoris sequence* associated with the decomposition of K as the union of the subcomplexes L and M.

(b) Using the results that $H_2(L) = 0$, $H_2(M) = 0$ and $H_1(M) = 0$, we obtain the following exact sequence from the Mayer-Vietoris sequence

$$\begin{aligned} 0 \to H_2(K) &\xrightarrow{\alpha_2} H_1(L \cap M) \xrightarrow{i_*} H_1(L) \xrightarrow{u_*} H_1(K) \\ &\xrightarrow{\alpha_1} H_0(L \cap M) \xrightarrow{k_*} H_0(L) \cap H_0(M). \end{aligned}$$

Now $H_0(L \cap M) \cong H_0(L) \cong H_0(M) \cong \mathbb{Z}$, since $|L|$, $|M|$ and $|L \cap M|$ are connected. Moreover the homomorphisms $i_* H_0(L \cap M) \to H_0(L)$ and $j_*(L \cap M) \to H_0(M)$ induced by the relevant inclusion maps are isomorphisms, and $k_*(\gamma) = (i_*(\gamma), j_*(\gamma))$ for all $\gamma \in H_0(L \cap M)$. It follows that $k_*\colon H_0(L \cap M) \to H_0(L) \oplus H_0(M)$ is injective. It then follows from exactness that image $\alpha_1 = \ker k_* = \{0\}$, so that α_1 is the zero homomorphism. But then $\ker \alpha_1 =$

$H_1(K)$, and therefore (by exactness), $u_*\colon H_1(L) \to H_1(K)$ is surjective. We therefore obtain an exact sequence

$$0 \to H_2(K) \xrightarrow{\alpha_2} H_1(L \cap M) \xrightarrow{i_*} H_1(L) \xrightarrow{u_*} H_1(K) \to 0.$$

We now use the fact (given in the question) that there exists a 2-chain c of L such that $\partial_2 c = 2z_L - z_{L\cap M}$, where z_L is a 1-cycle of L whose homology class $[z_L]$ generates $H_1(L)$, and $z_{L\cap M}$ is a 1-cycle of $L \cap M$, whose homology class $[z_{L\cap M}]$ generates $H_1(L\cap M)$. But then

$$i_*[z_{L\cap M}] = [2z_L - \partial_2 c] = 2[z_L].$$

Thus $i_*\colon H_1(L \cap M) \to H_1(L)$ corresponds under appropriate isomorphisms to the homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ that sends each integer n to $2n$. We deduce that $\ker i_* = \{0\}$, and $H_1(L)/i_*(H_1(L\cap M)) \cong \mathbb{Z}/2\mathbb{Z}$. But $\alpha_2\colon H_2(K) \to H_1(L \cap M)$ is injective (by exactness), and $\alpha_2(H_2(K)) = \ker i_* = \{0\}$ (again by exactness). Therefore $H_2(K) = 0$. Also

$$H_1(K) = \operatorname{image} u_* \cong H_1(L)/\ker u_* = H_1(L)/i_*(H_1(L\cap M)) \cong \mathbb{Z}_2,$$

where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Finally we note that $H_0(K) \cong \mathbb{Z}$, since $|K|$ is connected. Thus

$$H_0(K) \cong \mathbb{Z}, \quad H_1(K) \cong \mathbb{Z}_2, \quad H_2(K) = 0,$$

(and $H_q(K) = 0$ when $q > 2$ and when $q < 0$).

9. (a) [Quoted from from printed lecture notes.] Two simplicial maps $s: K \to L$ and $t: K \to L$ between simplicial complexes K and L are said to be *contiguous* if, given any simplex σ of K, there exists a simplex τ of L such that $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ.

(b) [Quoted from from printed lecture notes.] Let $\mathbf{x}$ be a point in the interior of some simplex σ of K. Then $f(\mathbf{x})$ belongs to the interior of a unique simplex τ of L, and moreover $s(\mathbf{x}) \in \tau$ and $t(\mathbf{x}) \in \tau$, since s and t are simplicial approximations to the map f. But $s(\mathbf{x})$ and $t(\mathbf{x})$ are contained in the interior of the simplices $s(\sigma)$ and $t(\sigma)$ of L. It follows that $s(\sigma)$ and $t(\sigma)$ are faces of τ, and hence $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ, as required.

(c) [Quoted from from printed lecture notes.] Choose an ordering of the vertices of K. Then there are well-defined homomorphisms $D_q: C_q(K) \to C_{q+1}(L)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle.$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_q$ are the vertices of a q-simplex of K listed in increasing order (with respect to the chosen ordering of the vertices of K). Then

$$\partial_1(D_0(\langle \mathbf{v} \rangle)) = \partial_1(\langle s(\mathbf{v}), t(\mathbf{v}) \rangle) = \langle t(\mathbf{v}) \rangle - \langle s(\mathbf{v}) \rangle,$$

and thus $\partial_1 \circ D_0 = t_0 - s_0$. Also

$$\begin{aligned}
& D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q \rangle)) \\
&\quad = \sum_{i=0}^{q} (-1)^i D_{q-1}(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle) \\
&\quad = \sum_{i=0}^{q} \sum_{j=0}^{i-1} (-1)^{i+j} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q) \rangle \\
&\qquad + \sum_{i=0}^{q} \sum_{j=i+1}^{q} (-1)^{i+j-1} \langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle
\end{aligned}$$

and

$$\partial_{q+1}(D_q(\langle \mathbf{v}_0, \ldots \mathbf{v}_q \rangle))$$

$$\begin{aligned}
&= \sum_{j=0}^{q}(-1)^j \partial_{q+1}(\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle) \\
&= \sum_{j=0}^{q}\sum_{i=0}^{j-1}(-1)^{i+j}\langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle \\
&\quad +\langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q)\rangle + \sum_{j=1}^{q}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_{j-1}), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle \\
&\quad - \sum_{j=0}^{q-1}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_{j+1}), \ldots, t(\mathbf{v}_q)\rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q)\rangle \\
&\quad + \sum_{j=0}^{q}\sum_{i=j+1}^{q}(-1)^{i+j+1}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q)\rangle \\
&= -D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q\rangle)) + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q)\rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q)\rangle
\end{aligned}$$

and thus

$$\partial_{q+1} \circ D_q + D_{q-1} \circ \partial_q = t_q - s_q$$

for all $q > 0$. It follows that $t_q(z) - s_q(z) = \partial_{q+1}(D_q(z))$ for any q-cycle z of K, and therefore $s_*([z]) = t_*([z])$. Thus $s_* = t_*$ as homomorphisms from $H_q(K)$ to $H_q(L)$, as required.