Course 421: Algebraic Topology
Section 9: Introduction to Homological Algebra

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9 Introduction to Homological Algebra

9.1 Exact Sequences

In homological algebra we consider sequences

\[ \cdots \rightarrow F \xrightarrow{p} G \xrightarrow{q} H \rightarrow \cdots \]

where \( F, G, H \) etc. are modules over some unital ring \( R \) and \( p, q \) etc. are \( R \)-module homomorphisms. We denote the trivial module \( \{0\} \) by \( 0 \), and we denote by \( 0 \rightarrow G \) and \( G \rightarrow 0 \) the zero homomorphisms from \( 0 \) to \( G \) and from \( G \) to \( 0 \) respectively. (These zero homomorphisms are of course the only homomorphisms mapping out of and into the trivial module \( 0 \).)

Unless otherwise stated, all modules are considered to be left modules.

**Definition** Let \( R \) be a unital ring, let \( F, G \) and \( H \) be \( R \)-modules, and let \( p: F \rightarrow G \) and \( q: G \rightarrow H \) be \( R \)-module homomorphisms. The sequence \( F \xrightarrow{p} G \xrightarrow{q} H \) of modules and homomorphisms is said to be exact at \( G \) if and only if \( \text{image}(p: F \rightarrow G) = \ker(q: G \rightarrow H) \). A sequence of modules and homomorphisms is said to be exact if it is exact at each module occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A **monomorphism** is an injective homomorphism. An **epimorphism** is a surjective homomorphism. An **isomorphism** is a bijective homomorphism.

The following result follows directly from the relevant definitions.

**Lemma 9.1** Let \( R \) be a unital ring, and let \( h: G \rightarrow H \) be a homomorphism of \( R \)-modules. Then

- \( h: G \rightarrow H \) is a monomorphism if and only if \( 0 \rightarrow G \xrightarrow{h} H \) is an exact sequence;
- \( h: G \rightarrow H \) is an epimorphism if and only if \( G \xrightarrow{h} H \rightarrow 0 \) is an exact sequence;
- \( h: G \rightarrow H \) is an isomorphism if and only if \( 0 \rightarrow G \xrightarrow{h} H \rightarrow 0 \) is an exact sequence.

Let \( R \) be a unital ring, and let \( F \) be a submodule of an \( R \)-module \( G \). Then the sequence

\[ 0 \rightarrow F \xrightarrow{i} G \xrightarrow{q} G/F \rightarrow 0, \]
is exact, where \( G/F \) is the quotient module, \( i: F \hookrightarrow G \) is the inclusion homomorphism, and \( q: G \to G/F \) is the quotient homomorphism. Conversely, given any exact sequence of the form

\[
0 \to F \xrightarrow{i} G \xrightarrow{q} H \to 0,
\]

we can regard \( F \) as a submodule of \( G \) (on identifying \( F \) with \( i(F) \)), and then \( H \) is isomorphic to the quotient module \( G/F \). Exact sequences of this type are referred to as short exact sequences.

We now introduce the concept of a commutative diagram. This is a diagram depicting a collection of homomorphisms between various modules occurring on the diagram. The diagram is said to commute if, whenever there are two routes through the diagram from a module \( G \) to a module \( H \), the homomorphism from \( G \) to \( H \) obtained by forming the composition of the homomorphisms along one route in the diagram agrees with that obtained by composing the homomorphisms along the other route. Thus, for example, the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{p} & & \downarrow{q} \\
D & \xrightarrow{h} & E \\
\end{array}
\]

\[
\begin{array}{ccc}
 & & \downarrow{r} \\
 & & \downarrow{k} \\
 & & \downarrow{\psi} \\
H & \xrightarrow{\phi} & F
\end{array}
\]

.commutes if and only if \( q \circ f = h \circ p \) and \( r \circ g = k \circ q \).

**Proposition 9.2** Let \( R \) be a unital ring. Suppose that the following diagram of \( R \)-modules and \( R \)-module homomorphisms

\[
\begin{array}{cccccc}
G_1 & \xrightarrow{\theta_1} & G_2 & \xrightarrow{\theta_2} & G_3 & \xrightarrow{\theta_3} & G_4 & \xrightarrow{\theta_4} & G_5 \\
\downarrow{\psi_1} & & \downarrow{\psi_2} & & \downarrow{\psi_3} & & \downarrow{\psi_4} & & \downarrow{\psi_5} \\
H_1 & \xrightarrow{\phi_1} & H_2 & \xrightarrow{\phi_2} & H_3 & \xrightarrow{\phi_3} & H_4 & \xrightarrow{\phi_4} & H_5
\end{array}
\]

commutes and that both rows are exact sequences. Then the following results follow:

(i) if \( \psi_2 \) and \( \psi_4 \) are monomorphisms and if \( \psi_1 \) is a epimorphism then \( \psi_3 \) is an monomorphism,

(ii) if \( \psi_2 \) and \( \psi_4 \) are epimorphisms and if \( \psi_5 \) is a monomorphism then \( \psi_3 \) is an epimorphism.

**Proof** First we prove (i). Suppose that \( \psi_2 \) and \( \psi_4 \) are monomorphisms and that \( \psi_1 \) is an epimorphism. We wish to show that \( \psi_3 \) is a monomorphism. Let \( x \in G_3 \) be such that \( \psi_3(x) = 0 \). Then \( \psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0 \),
and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover
\[ \phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0, \]
hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since $\psi_1$ is an epimorphism. Then
\[ \psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y), \]
and hence $\theta_1(w) = y$, since $\psi_2$ is a monomorphism. But then
\[ x = \theta_2(y) = \theta_2(\theta_1(w)) = 0 \]
by exactness. Thus $\psi_3$ is a monomorphism.

Next we prove (ii). Thus suppose that $\psi_2$ and $\psi_4$ are epimorphisms and that $\psi_5$ is a monomorphism. We wish to show that $\psi_3$ is an epimorphism. Let $a$ be an element of $H_3$. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since $\psi_4$ is an epimorphism. Now
\[ \psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0, \]
hence $\theta_4(b) = 0$, since $\psi_5$ is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then
\[ \phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b), \]
hence $\phi_3(a - \psi_3(c)) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But $\psi_2$ is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then
\[ \psi_3(\theta_2(e)) = \phi_2(\psi_2(e)) = a - \psi_3(c). \]
Hence $a = \psi_3(c + \theta_2(e))$, and thus $a$ is in the image of $\psi_3$. This shows that $\psi_3$ is an epimorphism, as required.

The following result is an immediate corollary of Proposition 9.2.

**Lemma 9.3 (Five-Lemma)** Suppose that the rows of the commutative diagram of Proposition 9.2 are exact sequences and that $\psi_1$, $\psi_2$, $\psi_4$ and $\psi_5$ are isomorphisms. Then $\psi_3$ is also an isomorphism.
9.2 Chain Complexes

**Definition** A *chain complex* \( C_* \) is a (doubly infinite) sequence \((C_i : i \in \mathbb{Z})\) of modules over some unital ring, together with homomorphisms \( \partial_i : C_i \to C_{i-1} \) for each \( i \in \mathbb{Z} \), such that \( \partial_i \circ \partial_{i+1} = 0 \) for all integers \( i \).

The \( i \)th *homology group* \( H_i(C_*) \) of the complex \( C_* \) is defined to be the quotient group \( Z_i(C_*)/B_i(C_*) \), where \( Z_i(C_*) \) is the kernel of \( \partial_i : C_i \to C_{i-1} \) and \( B_i(C_*) \) is the image of \( \partial_{i+1} : C_{i+1} \to C_i \).

Note that if the modules \( C_* \) occurring in a chain complex \( C_* \) are modules over some unital ring \( R \) then the homology groups of the complex are also modules over this ring \( R \).

**Definition** Let \( C_* \) and \( D_* \) be chain complexes. A *chain map* \( f_* : C_* \to D_* \) is a sequence \( f_i : C_i \to D_i \) of homomorphisms which satisfy the commutativity condition \( \partial_i \circ f_i = f_{i-1} \circ \partial_i \) for all \( i \in \mathbb{Z} \).

Note that a collection of homomorphisms \( f_i : C_i \to D_i \) defines a chain map \( f_* : C_* \to D_* \) if and only if the diagram

\[
\begin{array}{ccccccc}
\cdots & \to & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} & \to & \cdots \\
\downarrow{f_{i+1}} & & \downarrow{f_i} & & \downarrow{f_{i-1}} & & \\
\cdots & \to & D_{i+1} & \xrightarrow{\partial_{i+1}} & D_i & \xrightarrow{\partial_i} & D_{i-1} & \to & \cdots
\end{array}
\]

is commutative.

Let \( C_* \) and \( D_* \) be chain complexes, and let \( f_* : C_* \to D_* \) be a chain map. Then \( f_i(Z_i(C_*)) \subset Z_i(D_*) \) and \( f_i(B_i(C_*)) \subset B_i(D_*) \) for all \( i \). It follows from this that \( f_i : C_i \to D_i \) induces a homomorphism \( f_* : H_i(C_*) \to H_i(D_*) \) of homology groups sending \([z]\) to \([f_i(z)]\) for all \( z \in Z_i(C_*), \) where \([z] = z + B_i(C_*), \) and \([f_i(z)] = f_i(z) + B_i(D_*). \)

**Definition** A *short exact sequence* \( 0 \to A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \to 0 \) of chain complexes consists of chain complexes \( A_* \), \( B_* \) and \( C_* \) and chain maps \( p_* : A_* \to B_* \) and \( q_* : B_* \to C_* \) such that the sequence \( 0 \to A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \to 0 \) is exact for each integer \( i \).
We see that $0 \longrightarrow A \xrightarrow{p} B \xrightarrow{q} C \longrightarrow 0$ is a short exact sequence of chain complexes if and only if the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & A_{i+1} \\
\downarrow \partial_{i+2} & & \downarrow \partial_{i+2} \\
A_i & \xrightarrow{p_i} & B_i \\
\downarrow \partial_i & & \downarrow \partial_i \\
0 & \longrightarrow & B_i \\
\downarrow \partial_i & & \downarrow \partial_i \\
0 & \longrightarrow & A_{i-1} \\
\downarrow \partial_{i-1} & & \downarrow \partial_{i-1} \\
\end{array}
\]

is a commutative diagram whose rows are exact sequences and whose columns are chain complexes.

**Lemma 9.4** Given any short exact sequence $0 \longrightarrow A \xrightarrow{p} B \xrightarrow{q} C \longrightarrow 0$ of chain complexes, there is a well-defined homomorphism

$$\alpha_i : H_i(C) \to H_{i-1}(A)$$

which sends the homology class $[z]$ of $z \in Z_i(C)$ to the homology class $[w]$ of any element $w$ of $Z_{i-1}(A)$ with the property that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$.

**Proof** Let $z \in Z_i(C)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i : B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$ 

But $p_{i-1} : A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element $w$ of $A_{i-1}$ such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2} : A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A)$. 

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Now let \( b, b' \in B_i \) satisfy \( q_i(b) = q_i(b') = z \), and let \( w, w' \in Z_{i-1}(A_* ) \) satisfy \( p_{i-1}(w) = \partial_i(b) \) and \( p_{i-1}(w') = \partial_i(b') \). Then \( q_i(b - b') = 0 \), and hence \( b' - b = p_i(a) \) for some \( a \in A_i \), by exactness. But then

\[
p_{i-1}(w + \partial_i(a)) = p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) = \partial_i(b') = p_{i-1}(w'),
\]

and \( p_{i-1}: A_{i-1} \to B_{i-1} \) is injective. Therefore \( w + \partial_i(a) = w' \), and hence \( [w] = [w'] \) in \( H_{i-1}(A_*) \). Thus there is a well-defined function \( \tilde{\alpha}_i: Z_i(C_*) \to H_{i-1}(A_*) \) which sends \( z \in Z_i(C_*) \) to \( [w] \in H_{i-1}(A_*) \), where \( w \in Z_{i-1}(A_*) \) is chosen such that \( p_{i-1}(w) = \partial_i(b) \) for some \( b \in B_i \) satisfying \( q_i(b) = z \). This function \( \tilde{\alpha}_i \) is clearly a homomorphism from \( Z_i(C_*) \) to \( H_{i-1}(A_*) \).

Suppose that elements \( z \) and \( z' \) of \( Z_i(C_*) \) represent the same homology class in \( H_i(C_*) \). Then \( z' = z + \partial_{i+1}c \) for some \( c \in C_{i+1} \). Moreover \( c = q_{i+1}(d) \) for some \( d \in B_{i+1} \), since \( q_{i+1}: B_{i+1} \to C_{i+1} \) is surjective. Choose \( b \in B_i \) such that \( q_i(b) = z \), and let \( b' = b + \partial_{i+1}(d) \). Then

\[
q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.
\]

Moreover \( \partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b) \) (since \( \partial_i \circ \partial_{i+1} = 0 \)). Therefore \( \tilde{\alpha}_i(z) = \tilde{\alpha}_i(z') \). It follows that the homomorphism \( \tilde{\alpha}_i: Z_i(C_*) \to H_{i-1}(A_*) \) induces a well-defined homomorphism \( \alpha_i: H_i(C_*) \to H_{i-1}(A_*) \), as required.

Let \( 0 \to A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \to 0 \) and \( 0 \to A'_* \xrightarrow{p'_*} B'_* \xrightarrow{q'_*} C'_* \to 0 \) be short exact sequences of chain complexes, and let \( \lambda_*: A_* \to A'_* \), \( \mu_*: B_* \to B'_* \) and \( \nu_*: C_* \to C'_* \) be chain maps. For each integer \( i \), let \( \alpha_i: H_i(C_*) \to H_{i-1}(A_*) \) and \( \alpha'_i: H_i(C'_*) \to H_{i-1}(A'_*) \) be the homomorphisms defined as described in Lemma 9.4. Suppose that the diagram

\[
\begin{array}{ccc}
0 & \to & A_* \\
\downarrow{\lambda_*} & & \downarrow{\mu_*} \\
0 & \to & A'_*
\end{array}
\quad \begin{array}{ccc}
1 & \to & B_* \\
\downarrow{\nu_*} & & \downarrow{\nu_*} \\
1 & \to & B'_*
\end{array}
\quad \begin{array}{ccc}
1 & \to & C_* \\
\downarrow{\lambda_*} & & \downarrow{\lambda_*} \\
1 & \to & C'_*
\end{array}
\to 0
\]

commutes (i.e., \( p'_i \circ \lambda_i = \mu_i \circ p_i \) and \( q'_i \circ \mu_i = \nu_i \circ q_i \) for all \( i \)). Then the square

\[
\begin{array}{ccc}
H_i(C_*) & \xrightarrow{\alpha_i} & H_{i-1}(A_*) \\
\downarrow{\nu_*} & & \downarrow{\lambda_*} \\
H_i(C'_*) & \xrightarrow{\alpha'_i} & H_{i-1}(A'_*)
\end{array}
\]

commutes for all \( i \in \mathbb{Z} \) (i.e., \( \lambda_* \circ \alpha_i = \alpha'_i \circ \nu_* \)).
Proposition 9.5 Let $0 \to A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \to 0$ be a short exact sequence of chain complexes. Then the (infinite) sequence

$$
\cdots \xrightarrow{\alpha_{i+1}} H_i(A_*) \xrightarrow{p_*} H_i(B_*) \xrightarrow{q_*} H_i(C_*) \xrightarrow{\alpha_i} H_{i-1}(A_*) \xrightarrow{p_*} H_{i-1}(B_*) \xrightarrow{q_*} \cdots
$$

of homology groups is exact, where $\alpha_i : H_i(C_*) \to H_{i-1}(A_*)$ is the well-defined homomorphism that sends the homology class $[z]$ of $z \in Z_i(C_*)$ to the homology class $[w]$ of any element $w \in Z_{i-1}(A_*)$ with the property that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$.

**Proof** First we prove exactness at $H_i(B_*)$. Now $q_i \circ p_i = 0$, and hence $q_i \circ p_i = 0$. Thus the image of $p_i : H_i(A_*) \to H_i(B_*)$ is contained in the kernel of $q_i : H_i(B_*) \to H_i(C_*)$. Let $x$ be an element of $Z_i(B_*)$ for which $[x] \in \ker q_i$. Then $q_i(x) = \partial_{i+1}(c)$ for some $c \in C_{i+1}$. But $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1} : B_{i+1} \to C_{i+1}$ is surjective. Then

$$q_i(x - \partial_{i+1}(d)) = q_i(x) - \partial_{i+1}(q_{i+1}(d)) = q_i(x) - \partial_{i+1}(c) = 0,$$

and hence $x - \partial_{i+1}(d) = p_i(a)$ for some $a \in A_i$, by exactness. Moreover

$$p_{i-1}(\partial_i(a)) = \partial_i(p_i(a)) = \partial_i(x - \partial_{i+1}(d)) = 0,$$

since $\partial_i(x) = 0$ and $\partial_i \circ \partial_{i+1} = 0$. But $p_{i-1} : A_{i-1} \to B_{i-1}$ is injective. Therefore $\partial_i(a) = 0$, and thus $a$ represents some element $[a]$ of $H_i(A_*)$. We deduce that

$$[x] = [x - \partial_{i+1}(d)] = [p_i(a)] = p_*([a]).$$

We conclude that the sequence of homology groups is exact at $H_i(B_*)$.

Next we prove exactness at $H_i(C_*)$. Let $x \in Z_i(B_*)$. Now

$$\alpha_i(q_*[x]) = \alpha_i([q_i(x)]) = [w],$$

where $w$ is the unique element of $Z_i(A_*)$ satisfying $p_{i-1}(w) = \partial_i(x)$. But $\partial_i(x) = 0$, and hence $w = 0$. Thus $\alpha_i \circ q_* = 0$. Now let $z$ be an element of $Z_i(C_*)$ for which $[z] \in \ker \alpha_i$. Choose $b \in B_i$ and $w \in Z_{i-1}(A_*)$ such that $q_i(b) = z$ and $p_{i-1}(w) = \partial_i(b)$. Then $w = \partial_i(a)$ for some $a \in A_i$, since $[w] = \alpha_i([z]) = 0$. But then $q_i(b - p_i(a)) = z$ and $\partial_i(b - p_i(a)) = 0$. Thus $b - p_i(a) \in Z_i(B_*)$ and $q_*(b - p_i(a)) = [z]$. We conclude that the sequence of homology groups is exact at $H_i(C_*)$.

Finally we prove exactness at $H_{i-1}(A_*)$. Let $z \in Z_i(C_*)$. Then $\alpha_i([z]) = [w]$, where $w \in Z_{i-1}(A_*)$ satisfies $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. But then $p_*(\alpha_i([z])) = [p_{i-1}(w)] = [\partial_i(b)] = 0$. Thus $p_* \circ \alpha_i = 0$. 

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Now let $w$ be an element of $Z_{i-1}(A_\ast)$ for which $[w] \in \ker p_i$. Then $[p_{i-1}(w)] = 0$ in $H_{i-1}(B_\ast)$, and hence $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$. But

$$\partial_i(q_i(b)) = q_{i-1}(\partial_i(b)) = q_{i-1}(p_{i-1}(w)) = 0.$$  

Therefore $[w] = \alpha_i([z])$, where $z = q_i(b)$. We conclude that the sequence of homology groups is exact at $H_{i-1}(A_\ast)$, as required.