Course 421: Algebraic Topology
Section 9: Introduction to Homological Algebra

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Contents

9 Introduction to Homological Algebra

9.1 Exact Sequences

In homological algebra we consider sequences

$$\cdots \longrightarrow F \xrightarrow{p} G \xrightarrow{q} H \xrightarrow{\cdots}$$

where F, G, H etc. are modules over some unital ring R and p, q etc. are R-module homomorphisms. We denote the trivial module $\{0\}$ by 0, and we denote by $0 \longrightarrow G$ and $G \longrightarrow 0$ the zero homomorphisms from 0 to G and from G to 0 respectively. (These zero homomorphisms are of course the only homomorphisms mapping out of and into the trivial module 0.)

Unless otherwise stated, all modules are considered to be left modules.

Definition Let R be a unital ring, let F, G and H be R-modules, and let $p: F \to G$ and $q: G \to H$ be R-module homomorphisms. The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of modules and homomorphisms is said to be *exact* at G if and only if $\mathrm{image}(p: F \to G) = \ker(q: G \to H)$. A sequence of modules and homomorphisms is said to be *exact* if it is exact at each module occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A *monomorphism* is an injective homomorphism. An *epimorphism* is a surjective homomorphism. An *isomorphism* is a bijective homomorphism.

The following result follows directly from the relevant definitions.

Lemma 9.1 *let R be a unital ring, and let $h: G \to H$ be a homomorphism of R-modules. Then*

- *$h: G \to H$ is a monomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H$ is an exact sequence;*
- *$h: G \to H$ is an epimorphism if and only if $G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence;*
- *$h: G \to H$ is an isomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence.*

Let R be a unital ring, and let F be a submodule of an R-module G. Then the sequence

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} G/F \longrightarrow 0,$$

is exact, where G/F is the quotient module, $i: F \hookrightarrow G$ is the inclusion homomorphism, and $q: G \to G/F$ is the quotient homomorphism. Conversely, given any exact sequence of the form

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} H \longrightarrow 0,$$

we can regard F as a submodule of G (on identifying F with $i(F)$), and then H is isomorphic to the quotient module G/F. Exact sequences of this type are referred to as *short exact sequences.*

We now introduce the concept of a *commutative diagram.* This is a diagram depicting a collection of homomorphisms between various modules occurring on the diagram. The diagram is said to *commute* if, whenever there are two routes through the diagram from a module G to a module H, the homomorphism from G to H obtained by forming the composition of the homomorphisms along one route in the diagram agrees with that obtained by composing the homomorphisms along the other route. Thus, for example, the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ D & \xrightarrow{h} & E & \xrightarrow{k} & F \end{array}$$

commutes if and only if $q \circ f = h \circ p$ and $r \circ g = k \circ q$.

Proposition 9.2 *Let R be a unital ring. Suppose that the following diagram of R-modules and R-module homomorphisms*

$$\begin{array}{ccccccccc} G_1 & \xrightarrow{\theta_1} & G_2 & \xrightarrow{\theta_2} & G_3 & \xrightarrow{\theta_3} & G_4 & \xrightarrow{\theta_4} & G_5 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_4} & & \downarrow{\scriptstyle \psi_5} \\ H_1 & \xrightarrow{\phi_1} & H_2 & \xrightarrow{\phi_2} & H_3 & \xrightarrow{\phi_3} & H_4 & \xrightarrow{\phi_4} & H_5 \end{array}$$

commutes and that both rows are exact sequences. Then the following results follow:

(i) *if ψ_2 and ψ_4 are monomorphisms and if ψ_1 is a epimorphism then ψ_3 is an monomorphism,*

(ii) *if ψ_2 and ψ_4 are epimorphisms and if ψ_5 is a monomorphism then ψ_3 is an epimorphism.*

Proof First we prove (i). Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is an epimorphism. We wish to show that ψ_3 is a monomorphism. Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2(\theta_1(w)) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

Next we prove (ii). Thus suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. We wish to show that ψ_3 is an epimorphism. Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b),$$

hence $\phi_3(a - \psi_3(c)) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then

$$\psi_3(\theta_2(e)) = \phi_2(\psi_2(e)) = a - \psi_3(c).$$

Hence $a = \psi_3(c + \theta_2(e))$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism, as required. ∎

The following result is an immediate corollary of Proposition 9.2.

Lemma 9.3 (Five-Lemma) *Suppose that the rows of the commutative diagram of Proposition 9.2 are exact sequences and that ψ_1, ψ_2, ψ_4 and ψ_5 are isomorphisms. Then ψ_3 is also an isomorphism.*

9.2 Chain Complexes

Definition A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of modules over some unital ring, together with homomorphisms $\partial_i \colon C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i \colon C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1} \colon C_{i+1} \to C_i$.

Note that if the modules C_* occuring in a chain complex C_* are modules over some unital ring R then the homology groups of the complex are also modules over this ring R.

Definition Let C_* and D_* be chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence $f_i \colon C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

Note that a collection of homomorphisms $f_i \colon C_i \to D_i$ defines a chain map $f_* \colon C_* \to D_*$ if and only if the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} & \longrightarrow \cdots \\
& \Big\downarrow f_{i+1} & & \Big\downarrow f_i & & \Big\downarrow f_{i-1} & \\
\cdots \longrightarrow & D_{i+1} & \xrightarrow{\partial_{i+1}} & D_i & \xrightarrow{\partial_i} & D_{i-1} & \longrightarrow \cdots
\end{array}$$

is commutative.

Let C_* and D_* be chain complexes, and let $f_* \colon C_* \to D_*$ be a chain map. Then $f_i(Z_i(C_*)) \subset Z_i(D_*)$ and $f_i(B_i(C_*)) \subset B_i(D_*)$ for all i. It follows from this that $f_i \colon C_i \to D_i$ induces a homomorphism $f_* \colon H_i(C_*) \to H_i(D_*)$ of homology groups sending $[z]$ to $[f_i(z)]$ for all $z \in Z_i(C_*)$, where $[z] = z + B_i(C_*)$, and $[f_i(z)] = f_i(z) + B_i(D_*)$.

Definition A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_* \colon A_* \to B_*$ and $q_* \colon B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

We see that $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ is a short exact sequence of chain complexes if and only if the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial_{i+2}} & & \downarrow{\scriptstyle \partial_{i+2}} & & \downarrow{\scriptstyle \partial_{i+2}} & & \\
0 & \longrightarrow & A_{i+1} & \xrightarrow{p_{i+1}} & B_{i+1} & \xrightarrow{q_{i+1}} & C_{i+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{i+1}} & & \downarrow{\scriptstyle \partial_{i+1}} & & \downarrow{\scriptstyle \partial_{i+1}} & & \\
0 & \longrightarrow & A_i & \xrightarrow{p_i} & B_i & \xrightarrow{q_i} & C_i & \longrightarrow & 0\,. \\
 & & \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} & & \\
0 & \longrightarrow & A_{i-1} & \xrightarrow{p_{i-1}} & B_{i-1} & \xrightarrow{q_{i-1}} & C_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{i-1}} & & \downarrow{\scriptstyle \partial_{i-1}} & & \downarrow{\scriptstyle \partial_{i-1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

is a commutative diagram whose rows are exact sequences and whose columns are chain complexes.

Lemma 9.4 *Given any short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes, there is a well-defined homomorphism*

$$\alpha_i : H_i(C_*) \to H_{i-1}(A_*)$$

which sends the homology class $[z]$ *of* $z \in Z_i(C_*)$ *to the homology class* $[w]$ *of any element* w *of* $Z_{i-1}(A_*)$ *with the property that* $p_{i-1}(w) = \partial_i(b)$ *for some* $b \in B_i$ *satisfying* $q_i(b) = z$.

Proof Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i : B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1} : A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2} : A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_i$, by exactness. But then

$$p_{i-1}(w + \partial_i(a)) = p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) = \partial_i(b') = p_{i-1}(w'),$$

and $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function $\tilde{\alpha}_i$ is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}\colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism $\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*)$, as required. ∎

Let $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ and $0 \longrightarrow A'_* \xrightarrow{p'_*} B'_* \xrightarrow{q'_*} C'_* \longrightarrow 0$ be short exact sequences of chain complexes, and let $\lambda_*\colon A_* \to A'_*$, $\mu_*\colon B_* \to B'_*$ and $\nu_*\colon C_* \to C'_*$ be chain maps. For each integer i, let $\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*)$ and $\alpha'_i\colon H_i(C'_*) \to H_{i-1}(A'_*)$ be the homomorphisms defined as described in Lemma 9.4. Suppose that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{p_*} & B_* & \xrightarrow{q_*} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_*} & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \nu_*} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{p'_*} & B'_* & \xrightarrow{q'_*} & C'_* & \longrightarrow & 0
\end{array}$$

commutes (i.e., $p'_i \circ \lambda_i = \mu_i \circ p_i$ and $q'_i \circ \mu_i = \nu_i \circ q_i$ for all i). Then the square

$$\begin{array}{ccc}
H_i(C_*) & \xrightarrow{\alpha_i} & H_{i-1}(A_*) \\
\downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle \lambda_*} \\
H_i(C'_*) & \xrightarrow{\alpha'_i} & H_{i-1}(A'_*)
\end{array}$$

commutes for all $i \in \mathbb{Z}$ (i.e., $\lambda_* \circ \alpha_i = \alpha'_i \circ \nu_*$).

Proposition 9.5 *Let* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *be a short exact sequence of chain complexes. Then the (infinite) sequence*

$$\cdots \xrightarrow{\alpha_{i+1}} H_i(A_*) \xrightarrow{p_*} H_i(B_*) \xrightarrow{q_*} H_i(C_*) \xrightarrow{\alpha_i} H_{i-1}(A_*) \xrightarrow{p_*} H_{i-1}(B_*) \xrightarrow{q_*} \cdots$$

of homology groups is exact, where $\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*)$ *is the well-defined homomorphism that sends the homology class* $[z]$ *of* $z \in Z_i(C_*)$ *to the homology class* $[w]$ *of any element* w *of* $Z_{i-1}(A_*)$ *with the property that* $p_{i-1}(w) = \partial_i(b)$ *for some* $b \in B_i$ *satisfying* $q_i(b) = z$.

Proof First we prove exactness at $H_i(B_*)$. Now $q_i \circ p_i = 0$, and hence $q_* \circ p_* = 0$. Thus the image of $p_*\colon H_i(A_*) \to H_i(B_*)$ is contained in the kernel of $q_*\colon H_i(B_*) \to H_i(C_*)$. Let x be an element of $Z_i(B_*)$ for which $[x] \in \ker q_*$. Then $q_i(x) = \partial_{i+1}(c)$ for some $c \in C_{i+1}$. But $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}\colon B_{i+1} \to C_{i+1}$ is surjective. Then

$$q_i(x - \partial_{i+1}(d)) = q_i(x) - \partial_{i+1}(q_{i+1}(d)) = q_i(x) - \partial_{i+1}(c) = 0,$$

and hence $x - \partial_{i+1}(d) = p_i(a)$ for some $a \in A_i$, by exactness. Moreover

$$p_{i-1}(\partial_i(a)) = \partial_i(p_i(a)) = \partial_i(x - \partial_{i+1}(d)) = 0,$$

since $\partial_i(x) = 0$ and $\partial_i \circ \partial_{i+1} = 0$. But $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective. Therefore $\partial_i(a) = 0$, and thus a represents some element $[a]$ of $H_i(A_*)$. We deduce that

$$[x] = [x - \partial_{i+1}(d)] = [p_i(a)] = p_*([a]).$$

We conclude that the sequence of homology groups is exact at $H_i(B_*)$.

Next we prove exactness at $H_i(C_*)$. Let $x \in Z_i(B_*)$. Now

$$\alpha_i(q_*[x]) = \alpha_i([q_i(x)]) = [w],$$

where w is the unique element of $Z_i(A_*)$ satisfying $p_{i-1}(w) = \partial_i(x)$. But $\partial_i(x) = 0$, and hence $w = 0$. Thus $\alpha_i \circ q_* = 0$. Now let z be an element of $Z_i(C_*)$ for which $[z] \in \ker \alpha_i$. Choose $b \in B_i$ and $w \in Z_{i-1}(A_*)$ such that $q_i(b) = z$ and $p_{i-1}(w) = \partial_i(b)$. Then $w = \partial_i(a)$ for some $a \in A_i$, since $[w] = \alpha_i([z]) = 0$. But then $q_i(b - p_i(a)) = z$ and $\partial_i(b - p_i(a)) = 0$. Thus $b - p_i(a) \in Z_i(B_*)$ and $q_*([b - p_i(a)]) = [z]$. We conclude that the sequence of homology groups is exact at $H_i(C_*)$.

Finally we prove exactness at $H_{i-1}(A_*)$. Let $z \in Z_i(C_*)$. Then $\alpha_i([z]) = [w]$, where $w \in Z_{i-1}(A_*)$ satisfies $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. But then $p_*(\alpha_i([z])) = [p_{i-1}(w)] = [\partial_i(b)] = 0$. Thus $p_* \circ \alpha_i = 0$.

Now let w be an element of $Z_{i-1}(A_*)$ for which $[w] \in \ker p_*$. Then $[p_{i-1}(w)] = 0$ in $H_{i-1}(B_*)$, and hence $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$. But

$$\partial_i(q_i(b)) = q_{i-1}(\partial_i(b)) = q_{i-1}(p_{i-1}(w)) = 0.$$

Therefore $[w] = \alpha_i([z])$, where $z = q_i(b)$. We conclude that the sequence of homology groups is exact at $H_{i-1}(A_*)$, as required. ∎