Course 421: Algebraic Topology
Section 8: Modules

David R. Wilkins

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8.1 Rings and Fields

Definition A ring consists of a set \( R \) on which are defined operations of addition and multiplication that satisfy the following properties:

- the ring is an Abelian group with respect to the operation of addition;
- the operation of multiplication on the ring is associative, and thus\( x(yz) = (xy)z \) for all elements \( x, y \) and \( z \) of the ring.
- the operations of addition and multiplication satisfy the Distributive Law, and thus\( x(y + z) = xy + xz \) and \((x + y)z = xz + yz \) for all elements \( x, y \) and \( z \) of the ring.

Lemma 8.1 Let \( R \) be a ring. Then \( x0 = 0 \) and \( 0x = 0 \) for all elements \( x \) of \( R \).

Proof The zero element 0 of \( R \) satisfies \( 0 + 0 = 0 \). Using the Distributive Law, we deduce that \( x0 + x0 = x(0 + 0) = x0 \) and \( 0x + 0x = (0 + 0)x = 0x \). Thus if we add \( -(x0) \) to both sides of the identity \( x0 + x0 = x0 \) we see that \( x0 = 0 \). Similarly if we add \( -(0x) \) to both sides of the identity \( 0x + 0x = 0x \) we see that \( 0x = 0 \).

Lemma 8.2 Let \( R \) be a ring. Then \( (-x)y = -(xy) \) and \( x(-y) = -(xy) \) for all elements \( x \) and \( y \) of \( R \).

Proof It follows from the Distributive Law that \( xy + (-x)y = (x + (-x))y = 0y = 0 \) and \( xy + x(-y) = x(y + (-y)) = x0 = 0 \). Therefore \( (-x)y = -(xy) \) and \( x(-y) = -(xy) \).

A subset \( S \) of a ring \( R \) is said to be a subring of \( R \) if \( 0 \in S \), \( a + b \in S \), \(-a \in S \) and \( ab \in S \) for all \( a, b \in S \).

A ring \( R \) is said to be commutative if \( xy = yx \) for all \( x, y \in R \). Not every ring is commutative: an example of a non-commutative ring is provided by the ring of \( n \times n \) matrices with real or complex coefficients when \( n > 1 \).

A ring \( R \) is said to be unital if it possesses a (necessarily unique) non-zero multiplicative identity element 1 satisfying \( 1x = x = x1 \) for all \( x \in R \).

Definition A unital commutative ring \( R \) is said to be an integral domain if the product of any two non-zero elements of \( R \) is itself non-zero.
Definition A field consists of a set on which are defined operations of addition and multiplication that satisfy the following properties:

- the field is an Abelian group with respect to the operation of addition;
- the non-zero elements of the field constitute an Abelian group with respect to the operation of multiplication;
- the operations of addition and multiplication satisfy the Distributive Law, and thus \( x(y + z) = xy + xz \) and \( (x + y)z = xz + yz \) for all elements \( x, y \) and \( z \) of the field.

An examination of the relevant definitions shows that a unital commutative ring \( R \) is a field if and only if, given any non-zero element \( x \) of \( R \), there exists an element \( x^{-1} \) of \( R \) such that \( xx^{-1} = 1 \). Moreover a ring \( R \) is a field if and only if the set of non-zero elements of \( R \) is an Abelian group with respect to the operation of multiplication.

Lemma 8.3 A field is an integral domain.

Proof A field is a unital commutative ring. Let \( x \) and \( y \) be non-zero elements of a field \( K \). Then there exist elements \( x^{-1} \) and \( y^{-1} \) of \( K \) such that \( xx^{-1} = 1 \) and \( yy^{-1} = 1 \). Then \( xyy^{-1}x^{-1} = 1 \). It follows that \( xy \neq 0 \), since \( 0(y^{-1}x^{-1}) = 0 \) and \( 1 \neq 0 \).

The set \( \mathbb{Z} \) of integers is an integral domain with respect to the usual operations of addition and multiplication. The sets \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \) of rational, real and complex numbers are fields.

8.2 Modules

Definition Let \( R \) be a unital ring. A set \( M \) is said to be a left module over the ring \( R \) (or left \( R \)-module) if

- (i) given any \( x, y \in M \) and \( r \in R \), there are well-defined elements \( x + y \) and \( rx \) of \( M \),

- (ii) \( M \) is an Abelian group with respect to the operation + of addition,

- (iii) the identities

\[
r(x + y) = rx + ry, \quad (r + s)x = rx + sx,
\]

\[
(rs)x = r(sx), \quad 1_R x = x
\]

are satisfied for all \( x, y \in M \) and \( r, s \in R \), where \( 1_R \) denotes the multiplicative identity element of the ring \( R \).
Definition Let $R$ be a unital ring. A set $M$ is said to be a right module over $R$ (or right $R$-module) if

(i) given any $x, y \in M$ and $r \in R$, there are well-defined elements $x + y$ and $xr$ of $M$,

(ii) $M$ is an Abelian group with respect to the operation $+$ of addition,

(iii) the identities

\[(x + y)r = xr + yr, \quad x(r + s) = xr + xs,\]

\[x(rs) = (xr)s, \quad x1_R = x\]

are satisfied for all $x, y \in M$ and $r, s \in R$, where $1_R$ denotes the multiplicative identity element of the ring $R$.

If the unital ring $R$ is a commutative ring then there is no essential distinction between left $R$-modules and right $R$-modules. Indeed any left module $M$ over a unital commutative ring $R$ may be regarded as a right module on defining $xr = rx$ for all $x \in M$ and $r \in R$. We define a module over a unital commutative ring $R$ to be a left module over $R$.

Example If $K$ is a field, then a $K$-module is by definition a vector space over $K$.

Example Let $(M, +)$ be an Abelian group, and let $x \in M$. If $n$ is a positive integer then we define $nx$ to be the sum $x + x + \cdots + x$ of $n$ copies of $x$. If $n$ is a negative integer then we define $nx = -(|n|x)$, and we define $0x = 0$. This enables us to regard any Abelian group as a module over the ring $\mathbb{Z}$ of integers. Conversely, any module over $\mathbb{Z}$ is also an Abelian group.

Example Any unital commutative ring can be regarded as a module over itself in the obvious fashion.

Let $R$ be a unital ring that is not necessarily commutative, and let $+$ and $\times$ denote the operations of addition and multiplication defined on $R$. We denote by $R^{\text{op}}$ the ring $(R, +, \overline{\times})$, where the underlying set of $R^{\text{op}}$ is $R$ itself, the operation of addition on $R^{\text{op}}$ coincides with that on $R$, but where the operation of multiplication in the ring $R^{\text{op}}$ is the operation $\overline{\times}$ defined so that $r\overline{\times}s = s \times r$ for all $r, s \in R$. Note that the multiplication operation on the ring $R^{\text{op}}$ coincides with that on the ring $R$ if and only if the ring $R$ is commutative.
Any right module over the ring $R$ may be regarded as a left module over the ring $R^{op}$. Indeed let $M_R$ be a right $R$-module, and let $r.x = x$ for all $x \in M_R$ and $r \in R$. Then
\[ r.(s.x) = (s.x)r = x(sr) = x(rxs) = (rxs).x \]
for all $x \in M_R$ and $r, s \in R$. Also all other properties required of left modules over the ring $R^{op}$ are easily seen to be satisfied. It follows that any general results concerning left modules over unital rings yield corresponding results concerning right modules over unital rings.

Let $R$ be a unital ring, and let $M$ be a left $R$-module. A subset $L$ of $M$ is said to be a submodule of $M$ if $x + y \in L$ and $rx \in L$ for all $x, y \in L$ and $r \in R$. If $M$ is a left $R$-module and $L$ is a submodule of $M$ then the quotient group $M/L$ can itself be regarded as a left $R$-module, where $r(L+x) \equiv L+rx$ for all $L + x \in M/L$ and $r \in R$. The $R$-module $M/L$ is referred to as the quotient of the module $M$ by the submodule $L$.

A subset $L$ of a ring $R$ is said to be a left ideal of $R$ if $0 \in L$, $-x \in L$, $x + y \in L$ and $rx \in L$ for all $x, y \in L$ and $r \in R$. Any unital ring $R$ may be regarded as a left $R$-module, where multiplication on the left by elements of $R$ is defined in the obvious fashion using the multiplication operation on the ring $R$ itself. A subset of $R$ is then a submodule of $R$ (when $R$ is regarded as a left module over itself) if and only if this subset is a left ideal of $R$.

Let $M$ and $N$ be left modules over some unital ring $R$. A function $\varphi: M \to N$ is said to be a homomorphism of left $R$-modules if $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$ for all $x, y \in M$ and $r \in R$. A homomorphism of $R$-modules is said to be an isomorphism if it is invertible. The kernel $\ker \varphi$ and image $\varphi(M)$ of any homomorphism $\varphi: M \to N$ are themselves $R$-modules. Moreover if $\varphi: M \to N$ is a homomorphism of $R$-modules, and if $L$ is a submodule of $M$ satisfying $L \subset \ker \varphi$, then $\varphi$ induces a homomorphism $\varphi: M/L \to N$. This induced homomorphism is an isomorphism if and only if $L = \ker \varphi$ and $N = \varphi(M)$.

**Definition** Let $M_1, M_2, \ldots, M_k$ be left modules over a unital ring $R$. The direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ of the modules $M_1, M_2, \ldots, M_k$ is defined to be the set of ordered $k$-tuples $(x_1, x_2, \ldots, x_k)$, where $x_i \in M_i$ for $i = 1, 2, \ldots, k$. This direct sum is itself a left $R$-module, where
\[
(x_1, x_2, \ldots, x_k) + (y_1, y_2, \ldots, y_k) = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k),
\]
\[
r(x_1, x_2, \ldots, x_k) = (rx_1, rx_2, \ldots, rx_k)
\]
for all $x_i, y_i \in M_i$ and $r \in R$. 

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If $K$ is any field, then $K^n$ is the direct sum of $n$ copies of $K$.

**Definition** Let $M$ be a left module over some unital ring $R$. Given any subset $X$ of $M$, the submodule of $M$ generated by the set $X$ is defined to be the intersection of all submodules of $M$ that contain the set $X$. It is therefore the smallest submodule of $M$ that contains the set $X$. A left $R$-module $M$ is said to be *finitely-generated* if it is generated by some finite subset of itself.

**Lemma 8.4** Let $M$ be a left module over some unital ring $R$. Then the submodule of $M$ generated by some finite subset $\{x_1, x_2, \ldots, x_k\}$ of $M$ consists of all elements of $M$ that are of the form

$$ r_1 x_1 + r_2 x_2 + \cdots + r_k x_k $$

for some $r_1, r_2, \ldots, r_k \in R$.

**Proof** The subset of $M$ consisting of all elements of $M$ of this form is clearly a submodule of $M$. Moreover it is contained in every submodule of $M$ that contains the set $\{x_1, x_2, \ldots, x_k\}$. The result follows.

### 8.3 Bimodules

**Definition** Let $R$ and $S$ be unital rings. An $R$-$S$-*bimodule* is an Abelian group $M$, where elements of $M$ may be multiplied on the left by elements of $R$, and may also be multiplied on the right by elements of $S$, and where the following properties are satisfied:

(i) $M$ is a left $R$-module;

(ii) $M$ is a right $S$-module;

(iii) $(rx)s = r(xs)$ for all $x \in M$, $r \in R$ and $s \in S$.

**Example** Let $K$ be a field, let $m$ and $n$ be positive integers, and let $M_{m,n}(K)$ denote the set of $m \times n$ matrices with coefficients in the field $K$. Then $M_{m,n}(K)$ is an Abelian group with respect to the operation of matrix addition. The elements of $M_{m,n}(K)$ may be multiplied on the left by elements of the ring $M_m(K)$ of $m \times m$ matrices with coefficients in $K$; they may also be multiplied on the right by elements of the ring $M_n(K)$ of $n \times n$ matrices with coefficients in $K$; these multiplication operations are the usual ones resulting from matrix multiplication. Moreover $(AX)B = A(XB)$ for all $X \in M_{m,n}(K)$, $A \in M_m(K)$ and $B \in M_n(K)$. Thus $M_{m,n}(K)$ is an $M_m(K)$-$M_n(K)$-bimodule.
If \( R \) is a unital commutative ring then any \( R \)-module \( M \) may be regarded as an \( R\)-\( R \)-bimodule, where \((rx)s = r(xs) = (rs)x\) for all \( x \in M \) and \( r, s \in R \).

**Definition** Let \( R \) and \( S \) be unital rings, and let \( M \) and \( N \) be \( R \)-\( S \)-bimodules. A function \( \varphi: M \to N \) from \( M \) to \( N \) is said to be an \( R \)-\( S \)-bimodule homomorphism if \( \varphi(x + y) = \varphi(x) + \varphi(y) \), \( \varphi(rx) = r\varphi(x) \) and \( \varphi(xs) = \varphi(x)s \) for all \( x, y \in M \), \( r \in R \) and \( s \in S \).

### 8.4 Free Modules

**Definition** Let \( F \) be a left module over a unital ring \( R \), and let \( X \) be a subset of \( F \). We say that the left \( R \)-module \( F \) is freely generated by the subset \( X \) if, given any left \( R \)-module \( M \), and given any function \( f: X \to M \), there exists a unique \( R \)-module homomorphism \( \varphi: F \to M \) that extends the function \( f \).

**Example** Let \( K \) be a field. Then a \( K \)-module is a vector space over \( K \). Let \( V \) be a finite-dimensional vector space over the field \( K \), and let \( b_1, b_2, \ldots, b_n \) be a basis of \( V \). Then \( V \) is freely generated (as a \( K \)-module) by the set \( B \), where \( B = \{b_1, b_2, \ldots, b_n\} \). Indeed, given any vector space \( W \) over \( K \), and given any function \( f: B \to W \), there is a unique linear transformation \( \varphi: V \to W \) that extends \( f \). Indeed

\[
\varphi \left( \sum_{j=1}^{n} \lambda_j b_j \right) = \sum_{j=1}^{n} \lambda_j f(b_j)
\]

for all \( \lambda_1, \lambda_2, \ldots, \lambda_n \in K \). (Note that a function between vector spaces over some field \( K \) is a \( K \)-module homomorphism if and only if it is a linear transformation.)

**Definition** A left module \( F \) over a unital ring \( R \) is said to be free if there exists some subset of \( F \) that freely generates the \( R \)-module \( F \).

**Lemma 8.5** Let \( F \) be a left module over a unital ring \( R \), let \( X \) be a set, and let \( i: X \to F \) be a function. Suppose that the function \( i: X \to F \) satisfies the following universal property:

- given any left \( R \)-module \( M \), and given any function \( f: X \to M \), there exists a unique \( R \)-module homomorphism \( \varphi: F \to M \) such that \( \varphi \circ i = f \).

Then the function \( i: X \to F \) is injective, and \( F \) is freely generated by \( i(X) \).
**Proof** Let \( x \) and \( y \) be distinct elements of the set \( X \), and let \( f \) be a function satisfying \( f(x) = 0_R \) and \( f(y) = 1_R \), where \( 0_R \) and \( 1_R \) denote the zero element and the multiplicative identity element respectively of the ring \( R \). The ring \( R \) may be regarded as a left \( R \)-module over itself. It follows from the universal property of \( i: X \to M \) stated above that there exists a unique \( R \)-module homomorphism \( \theta: F \to R \) for which \( \theta \circ i = f \). Then \( \theta(i(x)) = 0_R \) and \( \theta(i(y)) = 1_R \). It follows that \( i(x) \neq i(y) \). Thus the function \( i: X \to F \) is injective.

Let \( M \) be a left \( R \)-module, and let \( g: i(X) \to M \) be a function defined on \( i(X) \). Then there exists a unique homomorphism \( \varphi: F \to M \) such that \( \varphi \circ i = g \circ i \). But then \( \varphi|_{i(X)} = g \). Thus the function \( g: i(X) \to M \) extends uniquely to a homomorphism \( \varphi: F \to M \). This shows that \( F \) is freely generated by \( i(X) \), as required.

Let \( F_1 \) and \( F_2 \) be left modules over a unital ring \( R \), let \( X_1 \) be a subset of \( F_1 \), and let \( X_2 \) be a subset of \( F_2 \). Suppose that \( F_1 \) is freely generated by \( X_1 \), and that \( F_2 \) is freely generated by \( X_2 \). Then any function \( f: X_1 \to X_2 \) from \( X_1 \) to \( X_2 \) extends uniquely to a \( R \)-module homomorphism from \( F_1 \) to \( F_2 \). We denote by \( f_2: F_1 \to F_2 \) the unique \( R \)-module homomorphism that extends \( f \).

Now let \( F_1, F_2 \) and \( F_3 \) be left modules over a unital ring \( R \), and let \( X_1, X_2 \) and \( X_3 \) be subsets of \( F_1, F_2 \) and \( F_3 \) respectively. Suppose that the left \( R \)-module \( F_i \) is freely generated by \( X_i \) for \( i = 1, 2, 3 \). Let \( f: X_1 \to X_2 \) and \( g: X_2 \to X_3 \) be functions. Then the functions \( f, g \) and \( g \circ f \) extend uniquely to \( R \)-module homomorphisms \( f_2: F_1 \to F_2, g_2: F_2 \to F_3 \) and \( (g \circ f)_2: F_3 \to F_3 \). Moreover the uniqueness of the homomorphism \( (g \circ f)_2 \) extending \( g \circ f \) suffices to ensure that \( (g \circ f)_2 = g_2 \circ f_2 \). Also the unique function from the module \( F_i \) extending the identity function of \( X_i \) is the identity isomorphism of \( F_i \), for each \( i \). It follows that if \( f: X_1 \to X_2 \) is a bijection, then \( f_2: F_1 \to F_2 \) is an isomorphism whose inverse is the unique homomorphism \( (f^{-1})_2: F_2 \to F_1 \) extending the inverse \( f^{-1}: X_2 \to X_1 \) of the bijection \( f \).

### 8.5 Construction of Free Modules

**Proposition 8.6** Let \( X \) be a set, and let \( R \) be a unital ring. Then there exists a left \( R \)-module \( F_R X \) and an injective function \( i_X: X \to F_R X \) such that \( F_R X \) is freely generated by \( i_X(X) \). The \( R \)-module \( F_R X \) and the function \( i_X: X \to F_R X \) then satisfy the following universal property:

- given any left \( R \)-module \( M \), and given any function \( f: X \to M \),
- there exists a unique \( R \)-module homomorphism \( \varphi: F_R X \to M \) such that \( \varphi \circ i_X = f \).

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The elements of $F_R X$ may be represented as functions from $X$ to $R$ that have only finitely many non-zero values. Also given any element $x$ of $X$, the corresponding element $i_X(x)$ of $F_R X$ is represented by the function $\delta_x: X \to R$, where $\delta_x$ maps $x$ to the identity element of $R$, and maps all other elements of $X$ to the zero element of $R$.

**Proof** Let $0_R$ and $1_R$ denote the zero element and the multiplicative identity element respectively of the ring $R$.

We define $F_R X$ to be the set of all functions $\sigma: X \to R$ from $X$ to $R$ that have at most finitely many non-zero values.

Note that if $\sigma$ and $\tau$ are functions from $X$ to $R$ that have at most finitely many non-zero values, then so is the sum $\sigma + \tau$ of the functions $\sigma$ and $\tau$ (where $(\sigma + \tau)(x) = \sigma(x) + \tau(x)$ for all $x \in X$). Therefore addition of functions is a binary operation on the set $F_R X$. Moreover $F_R X$ is an Abelian group with respect to the operation of addition of functions.

Given $r \in R$, and given $\sigma \in F_R X$, let $r\sigma$ be the function from $X$ to $R$ defined such that $(r\sigma)(x) = r\sigma(x)$ for all $x \in X$. Then

$$r(\sigma + \tau) = r\sigma + r\tau, \quad (r + s)\sigma = r\sigma + s\sigma,$$

$$(rs)\sigma = r(s\sigma), \quad 1_R \sigma = \sigma$$

for all $\sigma, \tau \in F_R X$ and $r, s \in R$. It follows that $F_R X$ is a module over the ring $R$.

Given $x \in X$, let $\delta_x: X \to R$ be the function defined such that $\delta_x(y) = \begin{cases} 1_R & \text{if } y = x; \\ 0_R & \text{if } y \neq x. \end{cases}$

Then $\delta_x \in F_R X$ for all $x \in X$. We denote by $i_X: X \to F_R X$ the function that sends $x$ to $\delta_x$ for all $x \in X$.

We claim that $F_R X$ is freely generated by the set $i_X(X)$, where $i_X(X) = \{ \delta_x : x \in X \}$. Let $M$ be an $R$-module, and let $f: X \to M$ be a function from $X$ to $M$. We must prove that there exists a unique $R$-module homomorphism $\varphi: F_R X \to M$ such that $\varphi \circ i_X = f$ (Lemma 8.5).

Let $\sigma$ be an element of $F_R X$. Then $\sigma$ is a function from $X$ to $R$ with at most finitely many non-zero values. Then $\sigma = \sum_{x \in \text{supp } \sigma} \sigma(x) \delta_x$, where

$$\text{supp } \sigma = \{ x \in X : \sigma(x) \neq 0_R \}.$$  

We define $\varphi(\sigma) = \sum_{x \in \text{supp } \sigma} \sigma(x)f(x)$. This associates to each element $\sigma$ of $F_R X$ a corresponding element $\varphi(\sigma)$ of $M$. We obtain in this way a function $\varphi: F_R X \to M$. 

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Let $\sigma$ and $\tau$ be elements of $F_RX$, let $r$ be an element of the ring $R$, and let $Y$ be a finite subset of $X$ for which $\text{supp}\sigma \subset Y$ and $\text{supp}\tau \subset Y$. Then 
\[ \text{supp}(\sigma + \tau) \subset Y, \]
and 
\[ \varphi(\sigma + \tau) = \sum_{x \in \text{supp}(\sigma + \tau)} (\sigma(x) + \tau(x))\delta_x = \sum_{x \in Y} (\sigma(x) + \tau(x))\delta_x = \sum_{x \in \text{supp}\sigma} \sigma(x)\delta_x + \sum_{x \in \text{supp}\tau} \tau(x)\delta_x = \varphi(\sigma) + \varphi(\tau). \]

Also 
\[ \varphi(r\sigma) = \sum_{x \in \text{supp}(r\sigma)} r\sigma(x)\delta_x = \sum_{x \in \text{supp}\sigma} r\sigma(x)\delta_x = r \left( \sum_{x \in \text{supp}\sigma} \sigma(x)\delta_x \right) = r\varphi(\sigma). \]

This shows that $\varphi: F_XR \to M$ is an $R$-module homomorphism. Moreover if $\psi: F_XR \to M$ is any $R$-module homomorphism satisfying $\psi \circ i_X = f$, then 
\[ \psi(\sigma) = \psi \left( \sum_{x \in \text{supp}\sigma} \sigma(x)\delta_x \right) = \sum_{x \in \text{supp}\sigma} \sigma(x)\psi(\delta_x) = \sum_{x \in \text{supp}\sigma} \sigma(x)\psi(i_X(x)) = \sum_{x \in \text{supp}\sigma} \sigma(x)f(x) = \varphi(\sigma). \]

Thus $\varphi: F_RX \to M$ is the unique $R$-module homomorphism satisfying $\varphi \circ i_X = f$.

It now follows from Lemma 8.5 that the $R$-module $F_RX$ is freely generated by $i_X(X)$. We have also shown that the required universal property is satisfied by the module $F_RX$ and the function $i_X$. 

**Definition** Let $X$ be a set, and let $R$ be a unital ring. We define the free left $R$-module on the set $X$ to be the module $F_RX$ constructed as described in the proof of Proposition 8.6. Moreover we may consider the set $X$ to be embedded in the free module $F_RX$ via the injective function $i_X: X \to F_XX$ described in the statement of that proposition.

Abelian groups are modules over the ring $\mathbb{Z}$ of integers. The construction of free modules therefore associates to any set $X$ a corresponding free Abelian group $F_\mathbb{Z}X$.

**Definition** Let $X$ be a set. The free Abelian group on the set $X$ is the module $F_\mathbb{Z}X$ whose elements can be represented as functions from $X$ to $\mathbb{Z}$ that have only finitely many non-zero values.
8.6 Tensor Products of Modules over a Unital Commutative Ring

**Definition** Let $R$ be a unital commutative ring, and let $M$ and $N$ and $P$ be $R$-modules. A function $f : M \times N \to P$ is said to be $R$-bilinear if

$$ f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y), $$

$$ f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2), $$

and

$$ f(rx, y) = rf(x, y) $$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$.

**Proposition 8.7** Let $R$ be a unital commutative ring, and let $M$ and $N$ be modules over $R$. Then there exists an $R$-module $M \otimes_R N$ and an $R$-bilinear function $j_{M \times N} : M \times N \to M \otimes_R N$, where $M \otimes_R N$ and $j_{M \times N}$ satisfy the following universal property:

given any $R$-module $P$, and given any $R$-bilinear function $f : M \times N \to P$, there exists a unique $R$-module homomorphism $\theta : M \otimes_R N \to P$ such that $f = \theta \circ j_{M \times N}$.

**Proof** Let $F_R(M \times N)$ be the free $R$-module on the set $M \times N$, and let $i_{M \times N} : M \times N \to F_R(M \times N)$ be the natural embedding of $M \times N$ in $F_R(M \times N)$. Then, given any $R$-module $P$, and given any function $f : M \times N \to P$, there exists a unique $R$-module homomorphism $\varphi : F_R(M \times N) \to P$ such that $\varphi \circ i_{M \times N} = f$ (Proposition 8.6).

Let $K$ be the submodule of $F_R(M \times N)$ generated by the elements

$$ i_{M \times N}(x_1 + x_2, y) - i_{M \times N}(x_1, y) - i_{M \times N}(x_2, y), $$

$$ i_{M \times N}(x, y_1 + y_2) - i_{M \times N}(x, y_1) - i_{M \times N}(x, y_2), $$

$$ i_{M \times N}(rx, y) - ri_{M \times N}(x, y), $$

$$ i_{M \times N}(x, ry) - ri_{M \times N}(x, y) $$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$. Also let $M \otimes_R N$ be the quotient module $F_R(M \times N)/K$, let $\pi : F_R(M \times N) \to M \otimes_R N$ be the quotient homomorphism, and let $j_{M \times N} : M \times N \to M \otimes_R N$ be the composition function $\pi \circ i_{M \times N}$. Then

$$ j_{M \times N}(x_1 + x_2, y) - j_{M \times N}(x_1, y) - j_{M \times N}(x_2, y) $$

$$ = \pi(i_{M \times N}(x_1 + x_2, y) - i_{M \times N}(x_1, y) - i_{M \times N}(x_2, y)) = 0 $$

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for all $x_1, x_2 \in M$ and $y \in N$. Similarly

$$j_{M \times N}(x, y_1 + y_2) - j_{M \times N}(x, y_1) - j_{M \times N}(x, y_2) = 0$$

for all $x \in M$ and $y_1, y_2 \in N$, and

$$j_{M \times N}(rx, y) - r j_{M \times N}(x, y) = \pi(i_{M \times N}(rx, y) - r i_{M \times N}(x, y)) = 0,$$

$$j_{M \times N}(x, ry) - r j_{M \times N}(x, y) = \pi(i_{M \times N}(x, ry) - r i_{M \times N}(x, y)) = 0$$

for all $x \in M$, $y \in N$ and $r \in R$. It follows that

$$j_{M \times N}(x_1 + x_2, y) = j_{M \times N}(x_1, y) + j_{M \times N}(x_2, y),$$

$$j_{M \times N}(x, y_1 + y_2) = j_{M \times N}(x, y_1) + j_{M \times N}(x, y_2),$$

and

$$j_{M \times N}(rx, y) = j_{M \times N}(x, ry) = r j_{M \times N}(x, y)$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$. Thus $j_{M \times N}: M \times N \to M \otimes_R N$ is an $R$-bilinear function.

Now let $P$ be an $R$-module, and let $f: M \times N \to P$ be an $R$-bilinear function. Then there is a unique $R$-module homomorphism $\varphi: F_R(M \times N) \to P$ such that $f = \varphi \circ i_{M \times N}$. Then

$$\varphi(i_{M \times N}(x_1 + x_2, y) - i_{M \times N}(x_1, y) - i_{M \times N}(x_2, y))$$

$$= f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y) = 0$$

for all $x_1, x_2 \in M$ and $y \in N$. Similarly

$$\varphi(i_{M \times N}(x, y_1 + y_2) - i_{M \times N}(x, y_1) - i_{M \times N}(x, y_2)) = 0$$

for all $x \in M$ and $y_1, y_2 \in N$, and

$$\varphi(i_{M \times N}(rx, y) - r i_{M \times N}(x, y)) = f(rx, y) - r f(x, y) = 0,$$

$$\varphi(i_{M \times N}(x, ry) - r i_{M \times N}(x, y)) = f(x, ry) - r f(x, y) = 0$$

for all $x \in M$, $y \in N$ and $r \in R$. Thus the submodule $K$ of $F_R(M \times N)$ is generated by elements of $\ker \varphi$, and therefore $K \subset \ker \varphi$. It follows that $\varphi: F_R(M \times N) \to P$ induces a unique $R$-module homomorphism $\theta: M \otimes_R N \to P$, where $M \otimes_R N = F_R(M \times N)/K$, such that $\varphi = \theta \circ \pi$. Then

$$\theta \circ j_{M \times N} = \theta \circ \pi \circ i_{M \times N} = \varphi \circ i_{M \times N} = f.$$ 

Moreover is $\psi: M \otimes_R N \to P$ is any $R$-module homomorphism satisfying $\psi \circ j_{M \times N} = f$ then $\psi \circ \pi \circ i_{M \times N} = f$. The uniqueness of the homomorphism $\varphi: F_R(M \times N) \to P$ then ensures that $\psi \circ \pi = \varphi = \theta \circ \pi$. But then $\psi = \theta$, because the quotient homomorphism $\pi: F_R(M \times N) \to M \otimes_R N$ is surjective. Thus the homomorphism $\theta$ is uniquely determined, as required. ■
Let $M$ and $N$ be modules over a unital commutative ring $R$. The module $M \otimes_R N$ constructed as described in the proof of Proposition 8.7 is referred to as the tensor product $M \otimes_R N$ of the modules $M$ and $N$ over the ring $R$. Given $x \in M$ and $y \in N$, we denote by $x \otimes y$ the image $j(x,y)$ of $(x,y)$ under the bilinear function $j_{M \times N}: M \times N \to M \otimes_R N$. We call this element the tensor product of the elements $x$ and $y$. Then
\[(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,\]
and
\[(rx) \otimes y = x \otimes (ry) = r(x \otimes y)\]
for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$. The universal property characterizing tensor products described in Proposition 8.7 then yields the following result.

**Corollary 8.8** Let $M$ and $N$ be modules over a unital commutative ring $R$, let $M \otimes_R N$ be the tensor product of $M$ and $N$ over $R$. Then, given any $R$-module $P$, and given any $R$-bilinear function $f: M \times N \to P$, there exists a unique $R$-module homomorphism $\theta: M \otimes_R N \to P$ such that $\theta(x \otimes y) = f(x,y)$ for all $x \in M$ and $y \in N$.

The following corollary shows that the universal property stated in Proposition 8.7 characterizes tensor products up to isomorphism.

**Corollary 8.9** Let $M$, $N$ and $T$ be modules over a unital commutative ring $R$, let $M \otimes_R N$ be the tensor product of $M$ and $N$, and let $k: M \times N \to T$ be an $R$-bilinear function. Suppose that $k: M \times N \to T$ satisfies the universal property characterizing tensor products so that, given any $R$-module $P$, and given any $R$-bilinear function $f: M \times N \to P$, there exists a unique $R$-module homomorphism $\psi: T \to P$ such that $f = \psi \circ k$. Then $T \cong M \otimes_R N$, and there is a unique $R$-isomorphism $\varphi: M \otimes_R N \to T$ such that $k(x,y) = \varphi(x \otimes y)$ for all $x \in M$ and $y \in N$.

**Proof** It follows from Corollary 8.8 that there exists a unique $R$-module homomorphism $\varphi: M \otimes_R N \to T$ such that $k(x,y) = \varphi(x \otimes y)$ for all $x \in M$ and $y \in N$. Also universal property satisfied by the bilinear function $k: M \times N \to T$ ensures that there exists a unique $R$-module homomorphism $\psi: T \to M \otimes_R N$ such that $x \otimes y = \psi(k(x,y))$ for all $x \in M$ and $y \in N$. Then $\psi(\varphi(x \otimes y)) = x \otimes y$ for all $x \in M$ and $y \in M$. But the universal property characterizing the tensor product ensures that any homomorphism from $M \times_R N$ to itself is determined uniquely by its action on elements of
the form \( x \otimes y \), where \( x \in M \) and \( y \in N \). It follows that \( \psi \circ \varphi \) is the identity automorphism of \( M \otimes_R N \). Similarly \( \varphi \circ \psi \) is the identity automorphism of \( T \).

It follows that \( \varphi: M \otimes_R N \to T \) is an isomorphism of \( R \)-modules whose inverse is \( \psi: T \to M \otimes_R N \). The isomorphism \( \varphi \) has the required properties.

**Corollary 8.10** Let \( M \) be a module over a unital commutative ring \( R \), and let \( \kappa: R \otimes_R M \to M \) be the \( R \)-module homomorphism defined such that \( \kappa(r \otimes x) = rx \) for all \( r \in R \) and \( x \in M \). Then \( \kappa \) is an isomorphism, and thus \( R \otimes_R M \cong M \).

**Proof** Let \( P \) be an \( R \)-module, and let \( f: R \times M \to P \) be an \( R \)-bilinear function. Let \( \psi: M \to P \) be defined such that \( \psi(x) = f(1_R, x) \) for all \( x \in M \), where \( 1_R \) denotes the identity element of the ring \( R \). Then \( \psi \) is an \( R \)-module homomorphism. Moreover \( f(r, x) = rf(1_R, x) = f(1_R, rx) = \psi(rx) \) for all \( x \in M \) and \( r \in R \). Thus \( f = \psi \circ k \), where \( k: R \times M \to M \) is the \( R \)-bilinear function defined such that \( k(r, x) = rx \) for all \( r \in R \) and \( x \in M \). The result therefore follows on applying Corollary 8.9.

**Corollary 8.11** Let \( M, M', N \) and \( N' \) be modules over a unital commutative ring \( R \), and let \( \varphi: M \to M' \) and \( \psi: N \to N' \) be \( R \)-module homomorphisms. Then \( \varphi \) and \( \psi \) induce an \( R \)-module homomorphism \( \varphi \otimes \psi: M \otimes_R N \to M' \otimes_R N' \), where \( (\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n) \) for all \( m \in M \) and \( n \in N \).

**Proof** The result follows immediately on applying Corollary 8.8 to the bilinear function from \( M \times N \) to \( M' \otimes_R N' \) that sends \( (m, n) \) to \( \varphi(m) \otimes \psi(n) \) for all \( m \in M \) and \( n \in N \).

### 8.7 Direct Sums and Tensor Products

**Lemma 8.12** Let \( L, M \) and \( N \) be \( R \)-modules over a unital commutative ring \( R \). Then

\[
(L \oplus M) \otimes_R N \cong (L \otimes_R N) \oplus (M \otimes_R N).
\]

**Proof** The function

\[
j: (L \oplus M) \times N \to (L \otimes_R N) \oplus (M \otimes_R N)
\]

is an \( R \)-bilinear function, where \( j((x, y), z) = (x \otimes z, y \otimes z) \) for all \( x \in L \), \( y \in M \) and \( z \in N \). We prove that the \( R \)-module \((L \otimes_R N) \oplus (M \otimes_R N)\) and the \( R \)-bilinear function \( j \) satisfy the universal property that characterizes the tensor product of \((L \oplus M)\) and \( N \) over the ring \( R \) up to isomorphism.
Let $P$ be an $R$-module, and let $f: (L \oplus M) \times N \to P$ be an $R$-bilinear function. Then $f$ determines $R$-bilinear functions $g: L \times N \to P$ and $h: M \times N \to P$, where $g(x, z) = f((x, 0), z)$ and $h(y, z) = f((0, x), z)$ for all $x \in L$, $y \in M$ and $z \in N$. Moreover

$$f((x, y), z) = f((x, 0)+(0, y), z) = f((x, 0), z)+f(0, y), z) = g(x, z)+h(y, z).$$

for all $x \in L$, $y \in M$ and $z \in N$. Now there exist unique $R$-module homomorphisms $\varphi: L \otimes_R N \to P$ $\psi: L \otimes_R N \to P$ satisfying the identities $\varphi(x \otimes z) = g(x, z)$ and $\psi(y \otimes z) = h(y, z)$ for all $x \in L$, $y \in M$ and $z \in N$. Then

$$f((x, y), z) = \varphi(x \otimes z) + \psi(y \otimes z) = \theta((x \otimes z), (y \otimes z)) = \theta(j((x, y), z),$$

where $\theta: (L \otimes_R N) \oplus (M \otimes_R N) \to P$ is the $R$-module homomorphism defined such that $\theta(u, v) = \varphi(u) + \psi(v)$ for all $u \in L \otimes_R N$ and $v \in M \otimes_R N$. We have thus shown that, given any $R$-module $P$, and given any $R$-bilinear function $f: (L \oplus M) \times N \to P$, there exists an $R$-module homomorphism $\theta: (L \otimes_R N) \oplus (M \otimes_R N) \to P$ satisfying $f = \text{theta} \circ j$. This homomorphism is uniquely determined. It follows directly from this that

$$(L \oplus M) \otimes_R N \cong (L \otimes_R N) \oplus (M \otimes_R N),$$

as required. 

8.8 Tensor Products of Abelian Groups

**Proposition 8.13** $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{\gcd(m,n)}$ for all positive integers $m$ and $n$, where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\gcd(m,n)$ is the greatest common divisor of $m$ and $n$.

**Proof** The cyclic groups $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are generated by $a$ and $b$ respectively, where $a = 1 + \mathbb{Z}_m$ and $b = 1 + \mathbb{Z}_n$. Moreover $\mathbb{Z}_m = \{ja : j \in \mathbb{Z}\}$, $\mathbb{Z}_n = \{kb : k \in \mathbb{Z}\}$, $j.a = 0$ if and only if $m$ divides the integer $j$, and $k.b = 0$ if and only if $n$ divides the integer $k$.

Now $\mathbb{Z}_m \otimes \mathbb{Z}_n$ is generated by elements of the form $x \otimes y$, where $x \in \mathbb{Z}_m$ and $y \in \mathbb{Z}_n$. Moreover $(j.a) \otimes (k.b) = jk(a \otimes b)$ for all integers $j$ and $k$. It follows that $\mathbb{Z}_m \otimes \mathbb{Z}_n = \{ja \otimes b : j \in \mathbb{Z}\}$. Thus the tensor product $\mathbb{Z}_m \otimes \mathbb{Z}_n$ is a cyclic group generated by $a \otimes b$. We must show that the order of this generator is the greatest common divisor of $m$ and $n$.

Let $r = \gcd(m,n)$. It follows from a basic result of elementary number theory that there exist integers $s$ and $t$ such that $r = sm + tn$. Then

$$r(a \otimes b) = sm(a \otimes b) + tn(a \otimes b) = s((ma) \otimes b) + t(a \otimes (nb)) = s(0 \otimes b) + t(a \otimes 0) = 0.$$
It follows that the generator \( a \otimes b \) of \( \mathbb{Z}_m \otimes \mathbb{Z}_n \) is an element of finite order, and the order of this element divides \( r \).

It remains to show that \( a \otimes b \) is of order \( r \). Now if \( j, j', k \) and \( k' \) are integers, and if \( j.a = j'.a \) and \( k.b = k'.b \) then \( m \) divides \( j - j' \) and \( n \) divides \( k - k' \). But then the greatest common divisor \( r \) of \( m \) and \( n \) divides \( jk - j'k' \), since \( jk - j'k' = (j - j')k + j'(k - k') \). Let \( c \) be the generator \( 1 + rz \) of \( \mathbb{Z}_r \). Then \( \mathbb{Z}_r \) is a well-defined bilinear function \( f : \mathbb{Z}_m \times \mathbb{Z} \rightarrow \mathbb{Z}_r \), where \( f(j.a, k.b) = jk.c \) for all integers \( j \) and \( k \). This function induces a unique group homomorphism \( \varphi : \mathbb{Z}_m \otimes \mathbb{Z} \rightarrow \mathbb{Z}_r \), where \( \varphi(x \otimes y) = f(x, y) \) for all \( x \in \mathbb{Z}_m \) and \( y \in \mathbb{Z}_n \). Then \( \varphi(ja \otimes b) = jc \) for all integers \( j \). Now the generator \( c \) of \( \mathbb{Z}_r \) is of order \( r \), and thus \( jc = 0 \) only when \( r \) divides \( j \). It follows that \( ja \otimes b = 0 \) only when \( r \) divides \( j \). Thus the generator \( a \otimes j \) of \( \mathbb{Z}_m \otimes \mathbb{Z}_n \) is of order \( r \), and therefore \( \mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_r \), where \( r = \gcd(m, n) \), as required.

There is a fundamental theorem concerning the structure of finitely-generated Abelian groups, which asserts that any finitely-generated Abelian group is isomorphic to the direct sum of a finite number of cyclic groups. Thus, given any Abelian group \( A \), there exist positive integers \( n_1, n_2, \ldots, n_k \) and \( r \) such that

\[
A \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}^r.
\]

Now Corollary 8.10 ensures that \( \mathbb{Z} \otimes \mathbb{Z} B \cong B \) for any Abelian group \( B \). It follows from Lemma 8.12 that

\[
A \otimes \mathbb{Z} B \cong (\mathbb{Z}_{n_1} \otimes \mathbb{Z} B) \oplus (\mathbb{Z}_{n_2} \otimes \mathbb{Z} B) \oplus \cdots \oplus (\mathbb{Z}_{n_k} \otimes \mathbb{Z} B) \oplus \mathbb{Z}^r.
\]

On applying Proposition 8.13, we find in particular that

\[
A \otimes \mathbb{Z} \mathbb{Z}_m \cong \mathbb{Z}_{\gcd(n_1, m)} \oplus \mathbb{Z}_{\gcd(n_2, m)} \oplus \cdots \oplus \mathbb{Z}_{\gcd(n_k, m)} \oplus \mathbb{Z}_m^r
\]

for any positive integer \( r \). Also \( A \otimes \mathbb{Z} \mathbb{Z} \cong A \), by Corollary 8.10.

Note that that \( \mathbb{Z}_1 \) is the zero group 0, and therefore \( 0 \oplus B \cong B \) for any Abelian group. (Indeed \( 0 \times B = \{ (0, b) : b \in B \} \), and this group of ordered pairs of the form \( (0, b) \) with \( b \in B \) is obviously isomorphic to \( B \).) We are thus in a position to evaluate the tensor product of any two finitely-generated Abelian groups.

Note also that if integers \( m \) and \( n \) are coprime, then \( \mathbb{Z}_{mn} \cong \mathbb{Z}_m \oplus \mathbb{Z}_n \). Indeed let \( a \in \mathbb{Z}_m \) be an element of order \( m \) (which therefore generates \( \mathbb{Z}_m \)), and let \( b \in \mathbb{Z}_n \) be an element of order \( n \). Then the order of the element \( (a, b) \) of \( \mathbb{Z}_m \oplus \mathbb{Z}_n \) is divisible by both \( m \) and \( n \), and is therefore divisible by \( mn \). It then follows that \( (a, b) \) generates the group \( \mathbb{Z}_m \oplus \mathbb{Z}_n \), and this group is therefore isomorphic to \( \mathbb{Z}_{mn} \).
Example Let 
\[ A \cong \mathbb{Z}_{18} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \] and 
\[ B \cong \mathbb{Z}_9 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5. \]
Then
\[ A \otimes_{\mathbb{Z}} B \cong (A \otimes \mathbb{Z}_9) \oplus (A \otimes \mathbb{Z}_4) \oplus A^5 \]
\[ \cong \mathbb{Z}_9 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^5 \oplus \mathbb{Z}_{18}^5 \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{10} \]
Now \( \mathbb{Z}_{18} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_9 \), because the integers 2 and 9 are coprime. (See remarks above). It follows that
\[ A \otimes_{\mathbb{Z}} B \cong \mathbb{Z}_6^2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_5^8 \oplus \mathbb{Z}_8^9 \oplus \mathbb{Z}_{10}^{10}. \]

8.9 Multilinear Maps and Tensor Products

Let \( M_1, M_2, \ldots, M_n \) be modules over a unital commutative ring \( R \), and let \( P \) be an \( R \)-module. A function \( f: M_1 \times M_2 \times \cdots \times M_n \to P \) is said to be \( R \)-multilinear if
\[
f(x_1, \ldots, x_{k-1}, x'_k + x''_k, x_{k+1}, \ldots, x_n)
= f(x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n)
+ f(x_1, \ldots, x_{k-1}, x''_k, x_{k+1}, \ldots, x_n)
\]
and
\[
f(x_1, \ldots, x_{k-1}, rx_k, x_{k+1}, \ldots, x_n) = rf(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)
\]
for \( k = 1, 2, \ldots, n \), for all \( x_l, x'_l, x''_l \in M_l \) \((l = 1, 2, \ldots, n)\), and for all \( r \in R \).
(When \( k = 1 \) the list \( x_1, \ldots, x_{k-1} \) should be interpreted as the empty list in the formulae above; when \( k = n \) the list \( x_{k+1}, \ldots, x_n \) should be interpreted as the empty list.) One can construct a module \( M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n \), referred to as the tensor product of the modules \( M_1, M_2, \ldots, M_n \) over the ring \( R \), and an \( R \)-multilinear mapping
\[
\tilde{j}_{M_1 \times M_2 \times \cdots \times M_n}: M_1 \times M_2 \times \cdots \times M_n \to M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n
\]
where the tensor product and multilinear mapping \( \tilde{j}_{M_1 \times M_2 \times \cdots \times M_n} \) satisfy the following universal property:

given any \( R \)-module \( P \), and given any \( R \)-multilinear function \( f: M_1 \times M_2 \times \cdots \times M_n \to P \), there exists a unique \( R \)-module homomorphism \( \theta: M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n \to P \) such that \( f = \theta \circ \tilde{j}_{M_1 \times M_2 \times \cdots \times M_n} \).
This tensor product is defined to be the quotient of the free module $F_R(M_1 \times M_2 \times \cdots \times M_n)$ by the submodule $K$ generated by elements of the free module that are of the form

$$i_{M_1 \times M_2 \times \cdots \times M_n}(x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n)$$

$$- i_{M_1 \times M_2 \times \cdots \times M_n}(x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n)$$

$$- i_{M_1 \times M_2 \times \cdots \times M_n}(x_1, \ldots, x_{k-1}, x''_k, x_{k+1}, \ldots, x_n),$$

or are of the form

$$i_{M_1 \times M_2 \times \cdots \times M_n}(x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n)$$

$$- ri_{M_1 \times M_2 \times \cdots \times M_n}(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n),$$

where $x_l, x'_l, x''_l \in M_l$ for $l = 1, 2, \ldots, n$, and $r \in R$. There is an $R$-multilinear function

$$j_{M_1 \times M_2 \times \cdots \times M_n}: M_1 \times M_2 \times \cdots \times M_n \to M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n,$$

where $j_{M_1 \times M_2 \times \cdots \times M_n}$ is the composition $\pi \circ i_{M_1 \times M_2 \times \cdots \times M_n}$ of the natural embedding

$$i_{M_1 \times M_2 \times \cdots \times M_n}: M_1 \times M_2 \times \cdots \times M_n \to F_R(M_1 \times M_2 \times \cdots \times M_n)$$

and the quotient homomorphism

$$\pi: F_R(M_1 \times M_2 \times \cdots \times M_n) \to M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n.$$

### 8.10 Tensor Products over Non-Commutative Rings

Let $R$ be a unital ring that is not necessarily commutative, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. These modules are Abelian groups under the operation of addition, and Abelian groups are modules over the ring $\mathbb{Z}$ of integers. We can therefore form their tensor product $M \otimes_{\mathbb{Z}} N$. This tensor product is an Abelian group.

Let $K$ be the subgroup of $M \otimes_{\mathbb{Z}} N$ generated by the elements

$$(xr) \otimes_{\mathbb{Z}} y - x \otimes_{\mathbb{Z}} (ry)$$

for all $x \in M$, $y \in N$ and $r \in R$, where $x \otimes_{\mathbb{Z}} y$ denotes the tensor product of $x$ and $y$ in the ring $M \otimes_{\mathbb{Z}} N$. We define the tensor product $M \otimes_R N$ of the right $R$-module $M$ and the left $R$-module $N$ over the ring $R$ to be the quotient group $M \otimes_{\mathbb{Z}} N/K$. Given $x \in M$ and $y \in N$, let $x \otimes y$ denote the
image of \( x \otimes \mathbb{Z} y \) under the quotient homomorphism \( \pi: M \otimes \mathbb{Z} N \rightarrow M \otimes_R N \).

Then

\[
(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,
\]

and

\[
(xr) \otimes y = x \otimes (ry)
\]

for all \( x, x_1, x_2 \in M, \ y, y_1, y_2 \in N \) and \( r \in R \).

**Lemma 8.14** Let \( R \) be a unital ring, let \( M \) be a right \( R \)-module, and let \( N \) be a left \( R \)-module. Then the tensor product \( M \otimes_R N \) of \( M \) and \( N \) is an Abelian group that satisfies the following universal property:

**given any Abelian group** \( P \), and **given any \( \mathbb{Z} \)-bilinear function** \( f: M \times N \rightarrow P \) which satisfies

\[
f(xr, y) = f(x, ry)
\]

**for all** \( x \in M, \ y \in N \) **and** \( r \in R \), there exists a unique Abelian group homomorphism \( \varphi: M \otimes_R N \rightarrow P \) such that \( f(x, y) = \varphi(x \otimes y) \) **for all** \( x \in M \) **and** \( y \in N \).

8.11 Tensor Products of Bimodules

Let \( Q, R \) and \( S \) be unital rings, let \( M \) be a \( Q \)-\( R \)-bimodule, and let \( N \) be an \( R \)-\( S \)-bimodule. Then \( M \) is a right \( R \)-module and \( N \) is a left \( R \)-module. We can therefore form the tensor product \( M \otimes_R N \) of \( M \) and \( N \) over the ring \( R \). This tensor product is an Abelian group under the operation of addition.

Let \( q \in Q \) and \( r \in R \). The definition of bimodules ensures that \((qx)r = q.xr\) for all \( x \in M \). Let \( L_q: M \times N \rightarrow M \otimes_R N \) be the function defined such that \( L_q(x, y) = (qx) \otimes y \) for all \( x \in M \) and \( y \in N \). Then the function \( f \) is \( \mathbb{Z} \)-bilinear. Moreover

\[
L_q(xr, y) = (q(xr)) \otimes y = ((qx)r) \otimes y = (qx) \otimes (ry) = L_q(x, ry).
\]

for all \( x \in M \) and \( y \in N \). It follows from Lemma 8.14 that there exists a group homomorphism \( \lambda_q: M \otimes_R N \rightarrow M \otimes_R N \), where \( \lambda_q(x \otimes y) = (qx) \otimes y \) for all \( x \in M \) and \( y \in N \). Similarly, given any element \( s \) of the ring \( S \), there exists a group homomorphism \( \rho_s: M \otimes_R N \rightarrow M \otimes_R N \), where \( \lambda_s(x \otimes y) = x \otimes (ys) \). We define \( qa = \lambda_q(a) \) and \( \alpha s = \rho_s(\alpha) \) for all \( \alpha \in M \otimes_R N \). One can check that \( M \otimes_R N \) is a \( Q \)-\( S \)-bimodule with respect to these operations of left multiplication by elements of \( Q \) and right multiplication by elements of
Moreover, given any $Q$-$S$-bimodule $P$, and given any $Z$-bilinear function $f: M \times N \to P$ that satisfies
\[ f(qx, y) = qf(x, y), \quad f(xr, y) = f(x, ry), \quad f(x, ys) = f(x, y)s \]
for all $x \in M$, $y \in N$, $q \in Q$, $r \in R$ and $s \in S$, there exists a unique $Q$-$S$ bimodule homomorphism $\varphi: M \otimes_R N \to P$ such that $f(x, y) = \varphi(x \otimes y)$ for all $x \in M$ and $y \in N$.

This construction generalizes the definition and universal property of the tensor product of modules over a unital commutative ring $R$, in view of the fact that any module over a unital commutative ring $R$ may be regarded as an $R$-$R$-bimodule.

### 8.12 Tensor Products involving Free Modules

**Proposition 8.15** Let $R$ and $S$ be unital rings, let $M$ be an $R$-$S$-bimodule and let $F_S X$ be a free left $S$-module on a set $X$. Then the tensor product $M \otimes_S F_S X$ is isomorphic, as an $R$-module, to $\Gamma(X, M)$, where $\Gamma(X, M)$ is the left $R$-module whose elements are represented as functions from $X$ to $M$ with only finitely many non-zero values, and where $(\lambda + \mu)(x) = \lambda(x) + \mu(x)$, and $(r\lambda)(x) = r\lambda(x)$ for all $\lambda, \mu \in \Gamma(X, M)$ and $r \in R$.

**Proof** The elements of the free left $S$-module $F_S X$ are represented as functions from $X$ to $S$. Let $f: M \times F_S X \to \Gamma(X, M)$ be the $Z$-bilinear function defined such that $f(m, \sigma)(x) = m\sigma(x)$ for all $m \in M$, $\sigma \in F_S X$ and $x \in X$. Then $f(ms, \sigma) = f(m, s\sigma)$ for all $m \in M$, $\sigma \in F_S X$ and $s \in S$. It follows from Lemma 8.14 that the function $f$ induces a unique homomorphism $\theta: M \otimes_S F_S X \to \Gamma(X, M)$ such that $\theta(m \otimes \sigma) = f(m, \sigma)$. Moreover $\theta$ is an $R$-module homomorphism.

Given $\mu \in \Gamma(X, M)$ we define
\[
\varphi(\mu) = \sum_{x \in \text{supp} \mu} \mu(x) \otimes \delta_x,
\]
where $\text{supp} \mu = \{x \in X : \mu(x) \neq 0\}$ and $\delta_x$ denotes the function from $X$ to $S$ which takes the value $1_S$ at $x$ and is zero elsewhere. Then $\varphi: \Gamma(X, M) \to M \otimes_S F_S X$ is also an $R$-module homomorphism. Now
\[
\varphi(\theta(m \otimes \sigma)) = \sum_{x \in \text{supp} \sigma} m\sigma(x) \otimes \delta_x = \sum_{x \in \text{supp} \sigma} m \otimes \sigma(x)\delta_x = m \otimes \left( \sum_{x \in \text{supp} \sigma} \sigma(x)\delta_x \right) = m \otimes \sigma
\]
for all $m \in M$ and $\sigma \in F_S X$. It follows that $\varphi \circ \theta$ is the identity automorphism of the tensor product $M \otimes_S F_S X$.

Also
\[
\theta(\varphi(\mu)) = \theta \left( \sum_{x \in \text{supp} \mu} \mu(x) \otimes \delta_x \right) = \sum_{x \in \text{supp} \mu} \theta(\mu(x) \otimes \delta_x)
\]
for all $\mu \in \Gamma(X, M)$. But
\[
\theta(\mu(x) \otimes \delta_x)(y) = \begin{cases} 
\mu(x) & \text{if } y = x; \\
0 & \text{if } y \neq x.
\end{cases}
\]
It follows that
\[
\theta(\varphi(\mu)) = \sum_{x \in \text{supp} \mu} \theta(\mu(x) \otimes \delta_x) = \mu
\]
for all $\mu \in \Gamma(X, M)$. Thus $\theta \circ \varphi$ is the identity automorphism of $\Gamma(X, M)$. We conclude that $\theta : M \otimes_S F_S X \to \Gamma(X, M)$ is an isomorphism of $R$-modules, as required.

Let $R$ be a unital ring. We can regard $R$ as an $R$-$Z$-bimodule, where $rn$ is the sum of $n$ copies of $r$ and $r(-n) = -rn$ for all non-negative integers $n$ and elements $r$ of $R$. We may therefore form the tensor product $R \otimes_Z A$ of the ring $R$ with any additive group $A$. (An additive group as an Abelian group where the group operation is expressed using additive notation.) This tensor product is an $R$-module. The following corollary is therefore a direct consequence of Proposition 8.15.

**Corollary 8.16** Let $R$ be a unital ring, let $X$ be a set, and let $F_Z X$ be the free Abelian group on the set $X$. Then $R \otimes_Z F_Z X \cong F_R X$. Thus the tensor product of the ring $R$ with any free Abelian group is a free $R$-module.

### 8.13 The Relationship between Bimodules and Left Modules

Let $R$ and $S$ be unital rings with multiplicative identity elements $1_R$ and $1_S$, and let $S^{\text{op}}$ be the unital ring $(S, +, \overline{\times})$ whose elements are those of $S$, whose operation of addition is the same as that defined on $S$, and whose operation $\overline{\times}$ of multiplication is defined such that $s_1 \overline{\times} s_2 = s_2 s_1$ for all $s_1, s_2 \in S$.

We can then construct a ring $R \otimes_Z S^{\text{op}}$. The elements of this ring belong to the tensor product of the rings $R$ and $S^{\text{op}}$ over the ring $Z$ of integers, and the operation of addition on $R \otimes_Z S^{\text{op}}$ is that defined on the tensor product. The operation of multiplication on $R \otimes_Z S^{\text{op}}$ is then defined such that
\[
(r_1 \otimes s_1) \times (r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 \overline{s_2}) = (r_1 r_2) \otimes (s_2 s_1).
\]
Lemma 8.17 Let $R$ and $S$ be unital rings, and let $M$ be an $R$-$S$-bimodule. Then $M$ is a left module over the ring $R \otimes_{\mathbb{Z}} S^{\text{op}}$, where

$$(r_1 \otimes s_1) \times (r_2 \otimes s_2) = (r_1 r_2) \otimes (s_2 s_1)$$

for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$, and where

$$(r \otimes s).x = (rx)s = r(xs)$$

for all $r \in R$, $s \in S$ and $x \in M$.

**Proof** Given any element $x$ of $M$, let $b_x : R \times S \to M$ be the function defined such that $b_x(r, s) = (rx)s = r(xs)$ for all $r \in R$ and $s \in S$. Then the function $b_x$ is $\mathbb{Z}$-bilinear, and therefore induces a unique $\mathbb{Z}$-module homomorphism $\beta_x : R \otimes_{\mathbb{Z}} S^{\text{op}} \to M$, where $\beta_x(r \otimes s) = b_x(r, s) = (rx)s$ for all $r \in R$, $s \in S$ and $x \in M$. We define $u.x = \beta_x(u)$ for all $u \in R \otimes_{\mathbb{Z}} S^{\text{op}}$ and $x \in M$. Then $(u_1 + u_2).x = u_1.x + u_2.x$ for all $u_1, u_2 \in R \otimes_{\mathbb{Z}} S^{\text{op}}$ and $x \in M$, because $\beta_x$ is a homomorphism of Abelian groups. Also $u.(x_1 + x_2) = u.x_1 + u.x_2$, because $b_{x_1 + x_2} = b_{x_1} + b_{x_2}$ and therefore $\beta_{x_1 + x_2} = \beta_{x_1} + \beta_{x_2}$.

Now

$$(r_1 \otimes s_1).((r_2 \otimes s_2).x) = (r_1 \otimes s_1).((r_2 x_2)s_2) = r_1(r_2(x_2s_2))s_1$$

$$= ((r_1r_2)(xs_2))s_1 = (r_1r_2)((xs_2)s_1)$$

$$= (r_1r_2)(x(s_2s_1)) = ((r_1r_2) \otimes_{\mathbb{Z}} (s_2s_1)).x$$

$$= ((r_1 \otimes_{\mathbb{Z}} s_1) \times (r_2 \otimes_{\mathbb{Z}} s_2)).x$$

for all $r_1, r_2 \in R$, $s_1, s_2 \in S$ and $x \in M$. The bilinearity of the function $\beta_x$ then ensures that $u_1.(u_2.x) = (u_1 \times u_2).x$ for all $u_1, u_2 \in R \otimes_{\mathbb{Z}} S^{\text{op}}$ and $x \in M$. Also $(1_R, 1_S).x = x$ for all $x \in M$, where $1_R$ and $1_S$ denote the identity elements of the rings $R$ and $S$. We conclude that $M$ is a left $R \otimes_{\mathbb{Z}} S^{\text{op}}$, as required. 

Let $R$ and $S$ be unital rings, and let $M$ be a left module over the ring $R \otimes_{\mathbb{Z}} S^{\text{op}}$. Then $M$ can be regarded as an $R$-$S$-bimodule, where $(rx)s = r(xs) = (r \otimes s).x$ for all $r \in R$, $s \in S$ and $x \in M$. We conclude therefore that all $R$-$S$-bimodules are left modules over the ring $R \otimes_{\mathbb{Z}} S^{\text{op}}$, and vice versa. It follows that any general result concerning left modules over unital rings yields a corresponding result concerning bimodules.