# Course 421: Algebraic Topology
# Section 8: Modules

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## Contents

# 8 Modules

## 8.1 Rings and Fields

**Definition** A *ring* consists of a set $R$ on which are defined operations of *addition* and *multiplication* that satisfy the following properties:

- the ring is an Abelian group with respect to the operation of addition;
- the operation of multiplication on the ring is associative, and thus $x(yz) = (xy)z$ for all elements $x$, $y$ and $z$ of the ring.
- the operations of addition and multiplication satisfy the *Distributive Law*, and thus $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$ for all elements $x$, $y$ and $z$ of the ring.

**Lemma 8.1** *Let $R$ be a ring. Then $x0 = 0$ and $0x = 0$ for all elements $x$ of $R$.*

**Proof** The zero element 0 of $R$ satisfies $0 + 0 = 0$. Using the Distributive Law, we deduce that $x0 + x0 = x(0 + 0) = x0$ and $0x + 0x = (0 + 0)x = 0x$. Thus if we add $-(x0)$ to both sides of the identity $x0 + x0 = x0$ we see that $x0 = 0$. Similarly if we add $-(0x)$ to both sides of the identity $0x + 0x = 0x$ we see that $0x = 0$. ∎

**Lemma 8.2** *Let $R$ be a ring. Then $(-x)y = -(xy)$ and $x(-y) = -(xy)$ for all elements $x$ and $y$ of $R$.*

**Proof** It follows from the Distributive Law that $xy+(-x)y = (x+(-x))y = 0y = 0$ and $xy + x(-y) = x(y + (-y)) = x0 = 0$. Therefore $(-x)y = -(xy)$ and $x(-y) = -(xy)$. ∎

A subset $S$ of a ring $R$ is said to be a *subring* of $R$ if $0 \in S$, $a + b \in S$, $-a \in S$ and $ab \in S$ for all $a, b \in S$.

A ring $R$ is said to be *commutative* if $xy = yx$ for all $x, y \in R$. Not every ring is commutative: an example of a non-commutative ring is provided by the ring of $n \times n$ matrices with real or complex coefficients when $n > 1$.

A ring $R$ is said to be *unital* if it possesses a (necessarily unique) non-zero multiplicative identity element 1 satisfying $1x = x = x1$ for all $x \in R$.

**Definition** A unital commutative ring $R$ is said to be an *integral domain* if the product of any two non-zero elements of $R$ is itself non-zero.

**Definition** A *field* consists of a set on which are defined operations of *addition* and *multiplication* that satisfy the following properties:

- the field is an Abelian group with respect to the operation of addition;
- the non-zero elements of the field constitute an Abelian group with respect to the operation of multiplication;
- the operations of addition and multiplication satisfy the *Distributive Law*, and thus $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$ for all elements $x$, $y$ and $z$ of the field.

An examination of the relevant definitions shows that a unital commutative ring $R$ is a field if and only if, given any non-zero element $x$ of $R$, there exists an element $x^{-1}$ of $R$ such that $xx^{-1} = 1$. Moreover a ring $R$ is a field if and only if the set of non-zero elements of $R$ is an Abelian group with respect to the operation of multiplication.

**Lemma 8.3** *A field is an integral domain.*

**Proof** A field is a unital commutative ring. Let $x$ and $y$ be non-zero elements of a field $K$. Then there exist elements $x^{-1}$ and $y^{-1}$ of $K$ such that $xx^{-1} = 1$ and $yy^{-1} = 1$. Then $xyy^{-1}x^{-1} = 1$. It follows that $xy \neq 0$, since $0(y^{-1}x^{-1}) = 0$ and $1 \neq 0$. ∎

The set $\mathbb{Z}$ of integers is an integral domain with respect to the usual operations of addition and multiplication. The sets $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ of rational, real and complex numbers are fields.

## 8.2 Modules

**Definition** Let $R$ be a unital ring. A set $M$ is said to be a *left module over the ring* $R$ (or *left* $R$*-module*) if

(i) given any $x, y \in M$ and $r \in R$, there are well-defined elements $x + y$ and $rx$ of $M$,

(ii) $M$ is an Abelian group with respect to the operation $+$ of addition,

(iii) the identities

$$r(x + y) = rx + ry, \qquad (r + s)x = rx + sx,$$

$$(rs)x = r(sx), \qquad 1_R x = x$$

are satisfied for all $x, y \in M$ and $r, s \in R$, where $1_R$ denotes the multiplicative identity element of the ring $R$.

**Definition** Let $R$ be a unital ring. A set $M$ is said to be a *right module over* $R$ (or *right* $R$*-module*) if

(i) given any $x, y \in M$ and $r \in R$, there are well-defined elements $x + y$ and $xr$ of $M$,

(ii) $M$ is an Abelian group with respect to the operation $+$ of addition,

(iii) the identities

$$(x + y)r = xr + yr, \qquad x(r + s) = xr + xs,$$

$$x(rs) = (xr)s, \qquad x1_R = x$$

are satisfied for all $x, y \in M$ and $r, s \in R$, where $1_R$ denotes the multiplicative identity element of the ring $R$.

If the unital ring $R$ is a commutative ring then there is no essential distinction between left $R$-modules and right $R$-modules. Indeed any left module $M$ over a unital commutative ring $R$ may be regarded as a right module on defining $xr = rx$ for all $x \in M$ and $r \in R$. We define a *module* over a unital commutative ring $R$ to be a left module over $R$.

**Example** If $K$ is a field, then a $K$-module is by definition a vector space over $K$.

**Example** Let $(M, +)$ be an Abelian group, and let $x \in M$. If $n$ is a positive integer then we define $nx$ to be the sum $x + x + \cdots + x$ of $n$ copies of $x$. If $n$ is a negative integer then we define $nx = -(|n|x)$, and we define $0x = 0$. This enables us to regard any Abelian group as a module over the ring $\mathbb{Z}$ of integers. Conversely, any module over $\mathbb{Z}$ is also an Abelian group.

**Example** Any unital commutative ring can be regarded as a module over itself in the obvious fashion.

Let $R$ be a unital ring that is not necessarily commutative, and let $+$ and $\times$ denote the operations of addition and multiplication defined on $R$. We denote by $R^{\mathrm{op}}$ the ring $(R, +, \overline{\times})$, where the underlying set of $R^{op}$ is $R$ itself, the operation of addition on $R^{op}$ coincides with that on $R$, but where the operation of multiplication in the ring $R^{\mathrm{op}}$ is the operation $\overline{\times}$ defined so that $r \overline{\times} s = s \times r$ for all $r, s \in R$. Note that the multiplication operation on the ring $R^{\mathrm{op}}$ coincides with that on the ring $R$ if and only if the ring $R$ is commutative.

Any right module over the ring $R$ may be regarded as a left module over the ring $R^{\mathrm{op}}$. Indeed let $M_R$ be a right $R$-module, and let $r.x = xr$ for all $x \in M_R$ and $r \in R$. Then

$$r.(s.x) = (s.x)r = x(sr) = x(r \overline{\times} s) = (r \overline{\times} s).x$$

for all $x \in M_R$ and $r, s \in R$. Also all other properties required of left modules over the ring $R^{\mathrm{op}}$ are easily seen to be satisfied. It follows that any general results concerning left modules over unital rings yield corresponding results concerning right modules over unital rings.

Let $R$ be a unital ring, and let $M$ be a left $R$-module. A subset $L$ of $M$ is said to be a *submodule* of $M$ if $x + y \in L$ and $rx \in L$ for all $x, y \in L$ and $r \in R$. If $M$ is a left $R$-module and $L$ is a submodule of $M$ then the quotient group $M/L$ can itself be regarded as a left $R$-module, where $r(L+x) \equiv L+rx$ for all $L + x \in M/L$ and $r \in R$. The $R$-module $M/L$ is referred to as the *quotient* of the module $M$ by the submodule $L$.

A subset $L$ of a ring $R$ is said to be a *left ideal* of $R$ if $0 \in L$, $-x \in L$, $x + y \in L$ and $rx \in L$ for all $x, y \in L$ and $r \in R$. Any unital ring $R$ may be regarded as a left $R$-module, where multiplication on the left by elements of $R$ is defined in the obvious fashion using the multiplication operation on the ring $R$ itself. A subset of $R$ is then a submodule of $R$ (when $R$ is regarded as a left module over itself) if and only if this subset is a left ideal of $R$.

Let $M$ and $N$ be left modules over some unital ring $R$. A function $\varphi: M \to N$ is said to be a *homomorphism of left $R$-modules* if $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$ for all $x, y \in M$ and $r \in R$. A homomorphism of $R$-modules is said to be an isomorphism if it is invertible. The kernel $\ker \varphi$ and image $\varphi(M)$ of any homomorphism $\varphi: M \to N$ are themselves $R$-modules. Moreover if $\varphi: M \to N$ is a homomorphism of $R$-modules, and if $L$ is a submodule of $M$ satisfying $L \subset \ker \varphi$, then $\varphi$ induces a homomorphism $\overline{\varphi}: M/L \to N$. This induced homomorphism is an isomorphism if and only if $L = \ker \varphi$ and $N = \varphi(M)$.

**Definition** Let $M_1, M_2, \ldots, M_k$ be left modules over a unital ring $R$. The *direct sum* $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ of the modules $M_1, M_2, \ldots, M_k$ is defined to be the set of ordered $k$-tuples $(x_1, x_2, \ldots, x_k)$, where $x_i \in M_i$ for $i = 1, 2, \ldots, k$. This direct sum is itself a left $R$-module, where

$$\begin{aligned} (x_1, x_2, \ldots, x_k) + (y_1, y_2, \ldots, y_k) &= (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k), \\ r(x_1, x_2, \ldots, x_k) &= (rx_1, rx_2, \ldots, rx_k) \end{aligned}$$

for all $x_i, y_i \in M_i$ and $r \in R$.

If $K$ is any field, then $K^n$ is the direct sum of $n$ copies of $K$.

**Definition** Let $M$ be a left module over some unital ring $R$. Given any subset $X$ of $M$, the submodule of $M$ generated by the set $X$ is defined to be the intersection of all submodules of $M$ that contain the set $X$. It is therefore the smallest submodule of $M$ that contains the set $X$. A left $R$-module $M$ is said to be *finitely-generated* if it is generated by some finite subset of itself.

**Lemma 8.4** *Let $M$ be a left module over some unital ring $R$. Then the submodule of $M$ generated by some finite subset $\{x_1, x_2, \ldots, x_k\}$ of $M$ consists of all elements of $M$ that are of the form*

$$r_1x_1 + r_2x_2 + \cdots + r_kx_k$$

*for some $r_1, r_2, \ldots, r_k \in R$.*

**Proof** The subset of $M$ consisting of all elements of $M$ of this form is clearly a submodule of $M$. Moreover it is contained in every submodule of $M$ that contains the set $\{x_1, x_2, \ldots, x_k\}$. The result follows. ∎

## 8.3 Bimodules

**Definition** Let $R$ and $S$ be unital rings. An *$R$-$S$-bimodule* is an Abelian group $M$, where elements of $M$ may be multiplied on the left by elements of $R$, and may also be multiplied on the right by elements of $S$, and where the following properties are satisfied:

(i) $M$ is a left $R$-module;

(ii) $M$ is a right $S$-module;

(iii) $(rx)s = r(xs)$ for all $x \in M$, $r \in R$ and $s \in S$.

**Example** Let $K$ be a field, let $m$ and $n$ be positive integers, and let $M_{m,n}(K)$ denote the set of $m \times n$ matrices with coefficients in the field $K$. Then $M_{m,n}(K)$ is an Abelian group with respect to the operation of matrix addition. The elements of $M_{m,n}(K)$ may be multiplied on the left by elements of the ring $M_m(K)$ of $m \times m$ matrices with coefficients in $K$; they may also be multiplied on the right by elements of the ring $M_n(K)$ of $n \times n$ matrices with coefficients in $K$; these multiplication operations are the usual ones resulting from matrix multiplication. Moreover $(AX)B = A(XB)$ for all $X \in M_{m,n}(K)$, $A \in M_m(K)$ and $B \in M_n(K)$. Thus $M_{m,n}(K)$ is an $M_m(K)$-$M_n(K)$-bimodule.

If $R$ is a unital commutative ring then any $R$-module $M$ may be regarded as an $R$-$R$-bimodule, where $(rx)s = r(xs) = (rs)x$ for all $x \in M$ and $r, s \in R$.

**Definition** Let $R$ and $S$ be unital rings, and let $M$ and $N$ be $R$-$S$-bimodules. A function $\varphi: M \to N$ from $M$ to $N$ is said to be an *$R$-$S$-bimodule homomorphism* if $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(rx) = r\varphi(x)$ and $\varphi(xs) = \varphi(x)s$ for all $x, y \in M$, $r \in R$ and $s \in S$.

## 8.4 Free Modules

**Definition** Let $F$ be a left module over a unital ring $R$, and let $X$ be a subset of $F$. We say that the left $R$-module $F$ is *freely generated* by the subset $X$ if, given any left $R$-module $M$, and given any function $f: X \to M$, there exists a unique $R$-module homomorphism $\varphi: F \to M$ that extends the function $f$.

**Example** Let $K$ be a field. Then a $K$-module is a vector space over $K$. Let $V$ be a finite-dimensional vector space over the field $K$, and let $b_1, b_2, \ldots, b_n$ be a basis of $V$. Then $V$ is freely generated (as a $K$-module) by the set $B$, where $B = \{b_1, b_2, \ldots, b_n\}$. Indeed, given any vector space $W$ over $K$, and given any function $f: B \to W$, there is a unique linear transformation $\varphi: V \to W$ that extends $f$. Indeed

$$\varphi\left(\sum_{j=1}^{n} \lambda_j b_j\right) = \sum_{j=1}^{n} \lambda_j f(b_j)$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in K$. (Note that a function between vector spaces over some field $K$ is a $K$-module homomorphism if and only if it is a linear transformation.)

**Definition** A left module $F$ over a unital ring $R$ is said to be *free* if there exists some subset of $F$ that freely generates the $R$-module $F$.

**Lemma 8.5** *Let $F$ be a left module over a unital ring $R$, let $X$ be a set, and let $i: X \to F$ be a function. Suppose that the function $i: X \to F$ satisfies the following* universal property*:*

> *given any left $R$-module $M$, and given any function $f: X \to M$, there exists a unique $R$-module homomorphism $\varphi: F \to M$ such that $\varphi \circ i = f$.*

*Then the function $i: X \to F$ is injective, and $F$ is freely generated by $i(X)$.*

**Proof** Let $x$ and $y$ be distinct elements of the set $X$, and let $f$ be a function satisfying $f(x) = 0_R$ and $f(y) = 1_R$, where $0_R$ and $1_R$ denote the zero element and the multiplicative identity element respectively of the ring $R$. The ring $R$ may be regarded as a left $R$-module over itself. It follows from the universal property of $i: X \to M$ stated above that there exists a unique $R$-module homomorphism $\theta: F \to R$ for which $\theta \circ i = f$. Then $\theta(i(x)) = 0_R$ and $\theta(i(y)) = 1_R$. It follows that $i(x) \neq i(y)$. Thus the function $i: X \to F$ is injective.

Let $M$ be a left $R$-module, and let $g: i(X) \to M$ be a function defined on $i(X)$. Then there exists a unique homomorphism $\varphi: F \to M$ such that $\varphi \circ i = g \circ i$. But then $\varphi|i(X) = g$. Thus the function $g: i(X) \to M$ extends uniquely to a homomorphism $\varphi: F \to M$. This shows that $F$ is freely generated by $i(X)$, as required. ∎

Let $F_1$ and $F_2$ be left modules over a unital ring $R$, let $X_1$ be a subset of $F_1$, and let $X_2$ be a subset of $F_2$. Suppose that $F_1$ is freely generated by $X_1$, and that $F_2$ is freely generated by $X_2$. Then any function $f: X_1 \to X_2$ from $X_1$ to $X_2$ extends uniquely to a $R$-module homomorphism from $F_1$ to $F_2$. We denote by $f_\sharp: F_1 \to F_2$ the unique $R$-module homomorphism that extends $f$.

Now let $F_1$, $F_2$ and $F_3$ be left modules over a unital ring $R$, and let $X_1$, $X_2$ and $X_3$ be subsets of $F_1$, $F_2$ and $F_3$ respectively. Suppose that the left $R$-module $F_i$ is freely generated by $X_i$ for $i = 1, 2, 3$. Let $f: X_1 \to X_2$ and $g: X_2 \to X_3$ be functions. Then the functions $f$, $g$ and $g \circ f$ extend uniquely to $R$-module homomorphisms $f_\sharp: F_1 \to F_2$, $g_\sharp: F_2 \to F_3$ and $(g \circ f)_\sharp: F_3 \to F_3$. Moreover the uniqueness of the homomorphism $(g \circ f)_\sharp$ extending $g \circ f$ suffices to ensure that $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. Also the unique function from the module $F_i$ extending the identity function of $X_i$ is the identity isomorphism of $F_i$, for each $i$. It follows that if $f: X_1 \to X_2$ is a bijection, then $f_\sharp: F_1 \to F_2$ is an isomorphism whose inverse is the unique homomorphism $(f^{-1})_\sharp: F_2 \to F_1$ extending the inverse $f^{-1}: X_2 \to X_1$ of the bijection $f$.

## 8.5 Construction of Free Modules

**Proposition 8.6** *Let $X$ be a set, and let $R$ be a unital ring. Then there exists a left $R$-module $F_R X$ and an injective function $i_X: X \to F_R X$ such that $F_R X$ is freely generated by $i_X(X)$. The $R$-module $F_R X$ and the function $i_X: X \to F_R X$ then satisfy the following* universal property*:*

> *given any left $R$-module $M$, and given any function $f: X \to M$, there exists a unique $R$-module homomorphism $\varphi: F_R X \to M$ such that $\varphi \circ i_X = f$.*

*The elements of $F_R X$ may be represented as functions from $X$ to $R$ that have only finitely many non-zero values. Also given any element $x$ of $X$, the corresponding element $i_X(x)$ of $F_R X$ is represented by the function $\delta_x\colon X \to R$, where $\delta_x$ maps $x$ to the identity element of $R$, and maps all other elements of $X$ to the zero element of $R$.*

**Proof** Let $0_R$ and $1_R$ denote the zero element and the multiplicative identity element respectively of the ring $R$.

We define $F_R X$ to be the set of all functions $\sigma\colon X \to R$ from $X$ to $R$ that have at most finitely many non-zero values.

Note that if $\sigma$ and $\tau$ are functions from $X$ to $R$ that have at most finitely many non-zero values, then so is the sum $\sigma+\tau$ of the functions $\sigma$ and $\tau$ (where $(\sigma+\tau)(x) = \sigma(x) + \tau(x)$ for all $x \in X$). Therefore addition of functions is a binary operation on the set $F_R X$. Moreover $F_R X$ is an Abelian group with respect to the operation of addition of functions.

Given $r \in R$, and given $\sigma \in F_R X$, let $r\sigma$ be the function from $X$ to $R$ defined such that $(r\sigma)(x) = r\sigma(x)$ for all $x \in X$. Then

$$r(\sigma + \tau) = r\sigma + r\tau, \qquad (r+s)\sigma = r\sigma + s\sigma,$$

$$(rs)\sigma = r(s\sigma), \qquad 1_R\sigma = \sigma$$

for all $\sigma, \tau \in F_R X$ and $r, s \in R$. It follows that $F_R X$ is a module over the ring $R$.

Given $x \in X$, let $\delta_x\colon X \to R$ be the function defined such that

$$\delta_x(y) = \begin{cases} 1_R & \text{if } y = x; \\ 0_R & \text{if } y \neq x. \end{cases}.$$

Then $\delta_x \in F_R X$ for all $x \in X$. We denote by $i_X\colon X \to F_R X$ the function that sends $x$ to $\delta_x$ for all $x \in X$.

We claim that $F_R X$ is freely generated by the set $i_X(X)$, where $i_X(X) = \{\delta_x : x \in X\}$. Let $M$ be an $R$-module, and let $f\colon X \to M$ be a function from $X$ to $M$. We must prove that there exists a unique $R$-module homomorphism $\varphi\colon F_R X \to M$ such that $\varphi \circ i_X = f$ (Lemma 8.5).

Let $\sigma$ be an element of $F_R X$. Then $\sigma$ is a function from $X$ to $R$ with at most finitely many non-zero values. Then $\sigma = \sum_{x \in \operatorname{supp} \sigma} \sigma(x)\delta_x$, where

$$\operatorname{supp} \sigma = \{x \in X : \sigma(x) \neq 0_R\}.$$

We define $\varphi(\sigma) = \sum_{x \in \operatorname{supp} \sigma} \sigma(x) f(x)$. This associates to each element $\sigma$ of $F_R X$ a corresponding element $\varphi(\sigma)$ of $M$. We obtain in this way a function $\varphi\colon F_R X \to M$.

Let $\sigma$ and $\tau$ be elements of $F_R X$, let $r$ be an element of the ring $R$, and let $Y$ be a finite subset of $X$ for which $\operatorname{supp}\sigma \subset Y$ and $\operatorname{supp}\tau \subset Y$. Then $\operatorname{supp}(\sigma+\tau) \subset Y$, and

$$\begin{aligned}\varphi(\sigma+\tau) &= \sum_{x\in\operatorname{supp}(\sigma+\tau)} (\sigma(x)+\tau(x))\delta_x = \sum_{x\in Y}(\sigma(x)+\tau(x))\delta_x \\ &= \sum_{x\in Y}\sigma(x)\delta_x + \sum_{x\in Y}\tau(x)\delta_x = \sum_{x\in\operatorname{supp}\sigma}\sigma(x)\delta_x + \sum_{x\in\operatorname{supp}\tau}\tau(x)\delta_x \\ &= \varphi(\sigma)+\varphi(\tau).\end{aligned}$$

Also

$$\varphi(r\sigma) = \sum_{x\in\operatorname{supp}(r\sigma)} r\sigma(x)\delta_x = \sum_{x\in\operatorname{supp}\sigma} r\sigma(x)\delta_x = r\left(\sum_{x\in\operatorname{supp}\sigma}\sigma(x)\delta_x\right) = r\varphi(\sigma).$$

This shows that $\varphi\colon F_X R \to M$ is an $R$-module homomorphism. Moreover if $\psi\colon F_X R \to M$ is any $R$-module homomorphism satisfying $\psi\circ i_X = f$, then

$$\begin{aligned}\psi(\sigma) &= \psi\left(\sum_{x\in\operatorname{supp}\sigma}\sigma(x)\delta_x\right) = \sum_{x\in\operatorname{supp}\sigma}\sigma(x)\psi(\delta_x) = \sum_{x\in\operatorname{supp}\sigma}\sigma(x)\psi(i_X(x)) \\ &= \sum_{x\in\operatorname{supp}\sigma}\sigma(x)f(x) = \varphi(\sigma).\end{aligned}$$

Thus $\varphi\colon F_R X \to M$ is the unique $R$-module homomorphism satisfying $\varphi\circ i_X = f$.

It now follows from Lemma 8.5 that the $R$-module $F_R X$ is freely generated by $i_X(X)$. We have also shown that the required universal property is satisfied by the module $F_R X$ and the function $i_X$. ∎

**Definition** Let $X$ be a set, and let $R$ be a unital ring. We define the *free left $R$-module* on the set $X$ to be the module $F_R X$ constructed as described in the proof of Proposition 8.6. Moreover we may consider the set $X$ to be embedded in the free module $F_R X$ via the injective function $i_X\colon X \to F_X X$ described in the statement of that proposition

Abelian groups are modules over the ring $\mathbb{Z}$ of integers. The construction of free modules therefore associates to any set $X$ a corresponding free Abelian group $F_{\mathbb{Z}} X$.

**Definition** Let $X$ be a set. The *free Abelian group* on the set $X$ is the module $F_{\mathbb{Z}} X$ whose elements can be represented as functions from $X$ to $\mathbb{Z}$ that have only finitely many non-zero values.

## 8.6 Tensor Products of Modules over a Unital Commutative Ring

**Definition** Let $R$ be a unital commutative ring, and let let $M$ and $N$ and $P$ be $R$-modules. A function $f\colon M \times N \to P$ is said to be $R$-bilinear if

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y),$$

$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2),$$

and

$$f(rx, y) = f(x, ry) = rf(x, y)$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$.

**Proposition 8.7** *Let $R$ be a unital commutative ring, and let $M$ and $N$ be modules over $R$. Then there exists an $R$-module $M \otimes_R N$ and an $R$-bilinear function $j_{M\times N}\colon M \times N \to M \otimes_R N$, where $M \otimes_R N$ and $j_{M\times N}$ satisfy the following universal property:*

> *given any $R$-module $P$, and given any $R$-bilinear function $f\colon M \times N \to P$, there exists a unique $R$-module homomorphism $\theta\colon M \otimes_R N \to P$ such that $f = \theta \circ j_{M\times N}$.*

**Proof** Let $F_R(M \times N)$ be the free $R$-module on the set $M \times N$, and let $i_{M\times N}\colon M \times N \to F_R(M \times N)$ be the natural embedding of $M \times N$ in $F_R(M \times N)$. Then, given any $R$-module $P$, and given any function $f\colon M \times N \to P$, there exists a unique $R$-module homomorphism $\varphi\colon F_R(M \times N) \to P$ such that $\varphi \circ i_{M\times N} = f$ (Proposition 8.6).

Let $K$ be the submodule of $F_R(M \times N)$ generated by the elements

$$i_{M\times N}(x_1 + x_2, y) - i_{M\times N}(x_1, y) - i_{M\times N}(x_2, y),$$

$$i_{M\times N}(x, y_1 + y_2) - i_{M\times N}(x, y_1) - i_{M\times N}(x, y_2),$$

$$i_{M\times N}(rx, y) - ri_{M\times N}(x, y),$$

$$i_{M\times N}(x, ry) - ri_{M\times N}(x, y)$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$. Also let $M \otimes_R N$ be the quotient module $F_R(M \times N)/K$, let $\pi\colon F_R(M \times N) \to M \otimes_R N$ be the quotient homomorphism, and let $j_{M\times N}\colon M \times N \to M \otimes_R N$ be the composition function $\pi \circ i_{M\times N}$. Then

$$\begin{aligned} & j_{M\times N}(x_1 + x_2, y) - j_{M\times N}(x_1, y) - j_{M\times N}(x_2, y) \\ &\quad = \pi(i_{M\times N}(x_1 + x_2, y) - i_{M\times N}(x_1, y) - i_{M\times N}(x_2, y)) = 0 \end{aligned}$$

for all $x_1, x_2 \in M$ and $y \in N$. Similarly

$$j_{M\times N}(x, y_1 + y_2, y) - j_{M\times N}(x, y_1) - j_{M\times N}(x, y_2) = 0$$

for all $x \in M$ and $y_1, y_2 \in N$, and

$$\begin{aligned} j_{M\times N}(rx, y) - rj_{M\times N}(x, y) &= \pi(i_{M\times N}(rx, y) - ri_{M\times N}(x, y)) = 0, \\ j_{M\times N}(x, ry) - rj_{M\times N}(x, y) &= \pi(i_{M\times N}(x, ry) - ri_{M\times N}(x, y)) = 0 \end{aligned}$$

for all $x \in M$, $y \in N$ and $r \in R$. It follows that

$$j_{M\times N}(x_1 + x_2, y) = j_{M\times N}(x_1, y) + j_{M\times N}(x_2, y),$$

$$j_{M\times N}(x, y_1 + y_2) = j_{M\times N}(x, y_1) + j_{M\times N}(x, y_2),$$

and

$$j_{M\times N}(rx, y) = j_{M\times N}(x, ry) = rj_{M\times N}(x, y)$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$. Thus $j_{M\times N}\colon M \times N \to M \otimes_R N$ is an $R$-bilinear function.

Now let $P$ be an $R$-module, and let $f\colon M \times N \to P$ be an $R$-bilinear function. Then there is a unique $R$-module homomorphism $\varphi\colon F_R(M \times N) \to P$ such that $f = \varphi \circ i_{M\times N}$. Then

$$\begin{aligned} &\varphi(i_{M\times N}(x_1 + x_2, y) - i_{M\times N}(x_1, y) - i_{M\times N}(x_2, y)) \\ &\qquad = f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y) = 0 \end{aligned}$$

for all $x_1, x_2 \in M$ and $y \in N$. Similarly

$$\varphi(i_{M\times N}(x, y_1 + y_2) - i_{M\times N}(x, y_1) - i_{M\times N}(x, y_2)) = 0$$

for all $x \in M$ and $y_1, y_2 \in N$, and

$$\varphi(i_{M\times N}(rx, y) - ri_{M\times N}(x, y)) = f(rx, y) - rf(x, y) = 0,$$

$$\varphi(i_{M\times N}(x, ry) - ri_{M\times N}(x, y)) = f(x, ry) - rf(x, y) = 0$$

for all $x \in M$, $y \in N$ and $r \in R$. Thus the submodule $K$ of $F_R(M \times N)$ is generated by elements of $\ker \varphi$, and therefore $K \subset \ker \varphi$. It follows that $\varphi\colon F_R(M \times N) \to P$ induces a unique $R$-module homomorphism $\theta\colon M \otimes_R N \to P$, where $M \otimes_R N = F_R(M \times N)/K$, such that $\varphi = \theta \circ \pi$. Then

$$\theta \circ j_{M\times N} = \theta \circ \pi \circ i_{M\times N} = \varphi \circ i_{M\times N} = f.$$

Moreover is $\psi\colon M \otimes_R N \to P$ is any $R$-module homomorphism satisfying $\psi \circ j_{M\times N} = f$ then $\psi \circ \pi \circ i_{M\times N} = f$. The uniqueness of the homomorphism $\varphi\colon F_R(M \times N) \to P$ then ensures that $\psi \circ \pi = \varphi = \theta \circ \pi$. But then $\psi = \theta$, because the quotient homomorphism $\pi\colon F_R(M \times N) \to M \otimes_R N$ is surjective. Thus the homomorphism $\theta$ is uniquely determined, as required. ∎

Let $M$ and $N$ be modules over a unital commutative ring $R$. The module $M \otimes_R N$ constructed as described in the proof of Proposition 8.7 is referred to as the *tensor product* $M \otimes_R N$ of the modules $M$ and $N$ over the ring $R$. Given $x \in M$ and $y \in N$, we denote by $x \otimes y$ the image $j(x, y)$ of $(x, y)$ under the bilinear function $j_{M\times N}: M \times N \to M \otimes_R N$. We call this element the *tensor product* of the elements $x$ and $y$. Then

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$$

and

$$(rx) \otimes y = x \otimes (ry) = r(x \otimes y)$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$. The universal property characterizing tensor products described in Proposition 8.7 then yields the following result.

**Corollary 8.8** *Let $M$ and $N$ be modules over a unital commutative ring $R$, let $M \otimes_R N$ be the tensor product of $M$ and $N$ over $R$. Then, given any $R$-module $P$, and given any $R$-bilinear function $f: M \times N \to P$, there exists a unique $R$-module homomorphism $\theta: M \otimes_R N \to P$ such that $\theta(x \otimes y) = f(x, y)$ for all $x \in M$ and $y \in N$.*

The following corollary shows that the universal property stated in Proposition 8.7 characterizes tensor products up to isomorphism.

**Corollary 8.9** *Let $M$, $N$ and $T$ be modules over a unital commutative ring $R$, let $M \otimes_R N$ be the tensor product of $M$ and $N$, and let $k: M \times N \to T$ be an $R$-bilinear function. Suppose that $k: M \times N \to T$ satisfies the universal property characterizing tensor products so that, given any $R$-module $P$, and given any $R$-bilinear function $f: M \times N \to P$, there exists a unique $R$-module homomorphism $\psi: T \to P$ such that $f = \psi \circ k$. Then $T \cong M \otimes_R N$, and there is a unique $R$-isomorphism $\varphi: M \otimes_R N \to T$ such that $k(x, y) = \varphi(x \otimes_R y)$ for all $x \in M$ and $y \in N$.*

**Proof** It follows from Corollary 8.8 that there exists a unique $R$-module homomorphism $\varphi: M \otimes_R N \to T$ such that $k(x, y) = \varphi(x \otimes y)$ for all $x \in M$ and $y \in N$. Also universal property satisfied by the bilinear function $k: M \times N \to T$ ensures that there exists a unique $R$-module homomorphism $\psi: T \to M \otimes_R N$ such that $x \otimes y = \psi(k(x, y))$ for all $x \in M$ and $y \in N$. Then $\psi(\varphi(x \otimes y)) = x \otimes y$ for all $x \in M$ and $y \in M$. But the universal property characterizing the tensor product ensures that any homomorphism from $M \times_R N$ to itself is determined uniquely by its action on elements of the form $x \otimes y$, where $x \in M$ and $y \in N$. It follows that $\psi \circ \varphi$ is the identity automorphism of $M \otimes_R N$. Similarly $\varphi \circ \psi$ is the identity automorphism of $T$. It follows that $\varphi: M \otimes_R N \to T$ is an isomorphism of $R$-modules whose inverse is $\psi: T \to M \otimes_R N$. The isomorphism $\varphi$ has the required properties. ∎

**Corollary 8.10** *Let $M$ be a module over a unital commutative ring $R$, and let $\kappa: R \otimes_R M \to M$ be the $R$-module homomorphism defined such that $\kappa(r \otimes x) = rx$ for all $r \in R$ and $x \in M$. Then $\kappa$ is an isomorphism, and thus $R \otimes_R M \cong M$.*

**Proof** Let $P$ be an $R$-module, and let $f: R \times M \to P$ be an $R$-bilinear function. Let $\psi: M \to P$ be defined such that $\psi(x) = f(1_R, x)$ for all $x \in M$, where $1_R$ denotes the identity element of the ring $R$. Then $\psi$ is an $R$-module homomorphism. Moreover $f(r, x) = rf(1_R, x) = f(1_R, rx) = \psi(rx)$ for all $x \in M$ and $r \in R$. Thus $f = \psi \circ k$, where $k: R \times M \to M$ is the $R$-bilinear function defined such that $k(r, x) = rx$ for all $r \in R$ and $x \in M$. The result therefore follows on applying Corollary 8.9. ∎

**Corollary 8.11** *Let $M$, $M'$, $N$ and $N'$ be modules over a unital commutative ring $R$, and let $\varphi: M \to M'$ and $\psi: N \to N'$ be $R$-module homomorphisms. Then $\varphi$ and $\psi$ induce an $R$-module homomorphism $\varphi \otimes \psi: M \otimes_R N \to M' \otimes_R N'$, where $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$ for all $m \in M$ and $n \in N$.*

**Proof** The result follows immediately on applying Corollary 8.8 to the bilinear function from $M \times N$ to $M' \otimes_R N'$ that sends $(m, n)$ to $\varphi(m) \otimes \psi(n)$ for all $m \in M$ and $n \in N$. ∎

## 8.7 Direct Sums and Tensor Products

**Lemma 8.12** *Let $L$, $M$ and $N$ be $R$-modules over a unital commutative ring $R$. Then*

$$(L \oplus M) \otimes_R N \cong (L \otimes_R N) \oplus (M \otimes_R N).$$

**Proof** The function

$$j: (L \oplus M) \times N \to (L \otimes_R N) \oplus (M \otimes_R N)$$

is an $R$-bilinear function, where $j((x, y), z) = (x \otimes z, y \otimes z)$ for all $x \in L$, $y \in M$ and $z \in N$. We prove that the $R$-module $(L \otimes_R N) \oplus (M \otimes_R N)$ and the $R$-bilinear function $j$ satisfy the universal property that characterizes the tensor product of $(L \oplus M)$ and $N$ over the ring $R$ up to isomorphism.

Let $P$ be an $R$-module, and let $f\colon (L \oplus M) \times N \to P$ be an $R$-bilinear function. Then $f$ determines $R$-bilinear functions $g\colon L \times N \to P$ and $h\colon M \times N \to P$, where $g(x,z) = f((x,0),z)$ and $h(y,z) = f((0,x),z$ for all $x \in L$, $y \in M$ and $z \in N$. Moreover

$$f((x,y),z) = f((x,0)+(0,y),z) = f((x,0),z)+f(0,y),z) = g(x,z)+h(y,z).$$

for all $x \in L$, $y \in M$ and $z \in N$. Now there exist unique $R$-module homomorphisms $\varphi\colon L \otimes_R N \to P$ $\psi\colon L \otimes_R N \to P$ satisfying the identities $\varphi(x \otimes z) = g(x,z)$ and $\psi(y \otimes z) = h(y,z)$ for all $x \in L$, $y \in M$ and $z \in N$. Then

$$f((x,y),z) = \varphi(x \otimes z) + \psi(y \otimes z) = \theta((x \otimes z),(y \otimes z)) = \theta(j((x,y),z),$$

where $\theta\colon (L \otimes_R N) \oplus (M \otimes_R N) \to P$ is the $R$-module homomorphism defined such that $\theta(u,v) = \varphi(u) + \psi(v)$ for all $u \in L \otimes_R N$ and $v \in M \otimes_R N$. We have thus shown that, given any $R$-module $P$, and given any $R$-bilinear function $f\colon (L \oplus M) \times N \to P$, there exists an $R$-module homomorphism $\theta\colon (L \otimes_R N) \oplus (M \otimes_R N) \to P$ satisfying $f = theta \circ j$. This homomorphism is uniquely determined. It follows directly from this that

$$(L \oplus M) \otimes_R N \cong (L \otimes_R N) \oplus (M \otimes_R N),$$

as required. ∎

## 8.8 Tensor Products of Abelian Groups

**Proposition 8.13** *$\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_{\gcd(m,n)}$ for all positive integers $m$ and $n$, where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\gcd(m,n)$ is the greatest common divisor of $m$ and $n$.*

**Proof** The cyclic groups $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are generated by $a$ and $b$ respectively, where $a = 1 + \mathbb{Z}_m$ and $b = 1 + \mathbb{Z}_n$. Moreover $\mathbb{Z}_m = \{j.a : j \in IZ\}$, $\mathbb{Z}_n = \{k.b : k \in \mathbb{Z}\}$, $j.a = 0$ if and only if $m$ divides the integer $j$, and $k.b = 0$ if and only if $n$ divides the integer $k$.

Now $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ is generated by elements of the form $x \otimes y$, where $x \in \mathbb{Z}_m$ and $y \in \mathbb{Z}_n$. Moreover $(j.a) \otimes (k.b) = jk(a \otimes b)$ for all integers $j$ and $k$. It follows that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \{ja \otimes b : j \in \mathbb{Z}\}$. Thus the tensor product $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ is a cyclic group generated by $a \otimes b$. We must show that the order of this generator is the greatest common divisor of $m$ and $n$.

Let $r = \gcd(m,n)$. It follows from a basic result of elementary number theory that there exist integers $s$ and $t$ such that $r = sm + tn$. Then

$$\begin{aligned} r(a \otimes b) &= sm(a \otimes b) + tn(a \otimes b) = s((ma) \otimes b) + t(a \otimes (nb)) \\ &= s(0 \otimes b) + t(a \otimes 0) = 0. \end{aligned}$$

It follows that the generator $a \otimes b$ of $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ is an element of finite order, and the order of this element divides $r$.

It remains to show that $a \otimes b$ is of order $r$. Now if $j$, $j'$, $k$ and $k'$ are integers, and if $j.a = j'.a$ and $k.b = k'.b$ then $m$ divides $j - j'$ and $n$ divides $k - k'$. But then the greatest common divisor $r$ of $m$ and $n$ divides $jk - j'k'$, since $jk - j'k' = (j - j')k + j'(k - k')$. Let $c$ be the generator $1 + r\mathbb{Z}$ of $\mathbb{Z}_r$. Then there is a well-defined bilinear function $f\colon \mathbb{Z}_m \times \mathbb{Z} \to \mathbb{Z}_r$, where $f(j.a, k.b) = jk.c$ for all integers $j$ and $k$. This function induces a unique group homomorphism $\varphi\colon \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}_r$, where $\varphi(x \otimes y) = f(x, y)$ for all $x \in \mathbb{Z}_m$ and $y \in \mathbb{Z}_n$. Then $\varphi(ja \otimes b) = jc$ for all integers $j$. Now the generator $c$ of $\mathbb{Z}_r$ is of order $r$, and thus $jc = 0$ only when $r$ divides $j$. It follows that $ja \otimes b = 0$ only when $r$ divides $j$. Thus the generator $a \otimes j$ of $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ is of order $r$, and therefore $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_r$, where $r = \gcd(m, n)$, as required. ∎

There is a fundamental theorem concerning the structure of finitely-generated Abelian groups, which asserts that any finitely-generated Abelian group is isomorphic to the direct sum of a finite number of cyclic groups. Thus, given any Abelian group $A$, there exist positive integers $n_1, n_2, \ldots, n_k$ and $r$ such that

$$A \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}^r.$$

Now Corollary 8.10 ensures that $\mathbb{Z} \otimes_{\mathbb{Z}} B \cong B$ for any Abelian group $B$. It follows from Lemma 8.12 that

$$A \otimes_{\mathbb{Z}} B \cong (\mathbb{Z}_{n_1} \otimes_{\mathbb{Z}} B) \oplus (\mathbb{Z}_{n_2} \otimes_{\mathbb{Z}} B) \oplus \cdots \oplus (\mathbb{Z}_{n_k} \otimes_{\mathbb{Z}} B) \oplus B^r.$$

On applying Proposition 8.13, we find in particular that

$$A \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_{\mathrm{gcd(n_1,m)}} \oplus \mathbb{Z}_{\mathrm{gcd(n_2,m)}} \oplus \cdots \oplus \mathbb{Z}_{\mathrm{gcd(n_k,m)}} \oplus \mathbb{Z}_m^r$$

for any positive integer $r$. Also $A \otimes_{\mathbb{Z}} \mathbb{Z} \cong A$, by Corollary 8.10.

Note that that $\mathbb{Z}_1$ is the zero group 0, and therefore $0 \oplus B \cong B$ for any Abelian group. (Indeed $0 \times B = \{(0, b) : b \in B\}$, and this group of ordered pairs of the form $(0, b)$ with $b \in B$ is obviously isomorphic to $B$.) We are thus in a position to evaluate the tensor product of any two finitely-generated Abelian groups

Note also that if integers $m$ and $n$ are coprime, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \oplus \mathbb{Z}_n$. Indeed let $a \in \mathbb{Z}_m$ be an element of order $m$ (which therefore generates $\mathbb{Z}_m$), and let $b \in \mathbb{Z}_n$ be an element of order $n$. Then the order of the element $(a, b)$ of $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is divisible by both $m$ and $n$, and is therefore divisible by $mn$. It then follows that $(a, b)$ generates the group $\mathbb{Z}_m \oplus \mathbb{Z}_n$, and this group is therefore isomorphic to $\mathbb{Z}_{mn}$.

**Example** Let

$$A \cong \mathbb{Z}_{18} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}^2 \text{ and } B \cong \mathbb{Z}_9 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^5.$$

Then

$$\begin{aligned} A \otimes_{\mathbb{Z}} B &\cong (A \otimes_{\mathbb{Z}} \mathbb{Z}_9) \oplus (A \otimes_{\mathbb{Z}} \mathbb{Z}_4) \oplus A^5 \\ &\cong \mathbb{Z}_9 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_9^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_{18}^5 \oplus \mathbb{Z}_8^5 \oplus \mathbb{Z}^{10} \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_8^5 \oplus \mathbb{Z}_9^3 \oplus \mathbb{Z}_{18}^5 \oplus \mathbb{Z}^{10}. \end{aligned}$$

Now $\mathbb{Z}_{18} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_9$, because the integers 2 and 9 are coprime. (See remarks above). It follows that

$$A \otimes_{\mathbb{Z}} B \cong \mathbb{Z}_2^6 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_8^5 \oplus \mathbb{Z}_9^8 \oplus \mathbb{Z}^{10}.$$

## 8.9 Multilinear Maps and Tensor Products

Let $M_1, M_2, \ldots, M_n$ be modules over a unital commutative ring $R$, and let $P$ be an $R$-module. A function $f\colon M_1 \times M_2 \times \cdots \times M_n \to P$ is said to be $R$-multilinear if

$$\begin{aligned} f(x_1, \ldots, x_{k-1}, x_k' + x_k'', x_{k+1}, \ldots, x_n) \\ = \; & f(x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_n) \\ & + f(x_1, \ldots, x_{k-1}, x_k'', x_{k+1}, \ldots, x_n) \end{aligned}$$

and

$$f(x_1, \ldots, x_{k-1}, rx_k, x_{k+1}, \ldots, x_n) = rf(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)$$

for $k = 1, 2, \ldots, n$, for all $x_l, x_l', x_l'' \in M_l$ $(l = 1, 2, \ldots, n)$, and for all $r \in R$. (When $k = 1$ the list $x_1, \ldots, x_{k-1}$ should be interpreted as the empty list in the formulae above; when $k = n$ the list $x_{k+1}, \ldots, x_n$ should be interpreted as the empty list.) One can construct a module $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$, referred to as the *tensor product* of the modules $M_1, M_2, \ldots, M_n$ over the ring $R$, and an $R$-multilinear mapping

$$j_{M_1 \times M_2 \times \cdots \times M_n}\colon M_1 \times M_2 \times \cdots \times M_n \to M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$$

where the tensor product and multilinear mapping $j_{M_1 \times M_2 \times \cdots \times M_n}$ satisfy the following universal property:

> given any $R$-module $P$, and given any $R$-multilinear function $f\colon M_1 \times M_2 \times \cdots \times M_n \to P$, there exists a unique $R$-module homomorphism $\theta\colon M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n \to P$ such that $f = \theta \circ j_{M_1 \times M_2 \times \cdots \times M_n}$.

This tensor product is defined to be the quotient of the free module $F_R(M_1 \times M_2 \times \cdots \times M_n)$ by the submodule $K$ generated by elements of the free module that are of the form

$$\begin{aligned} &i_{M_1\times M_2\times\cdots\times M_n}(x_1,\ldots,x_{k-1},x'_k+x''_k,x_{k+1},\ldots,x_n)\\ &\quad - i_{M_1\times M_2\times\cdots\times M_n}(x_1,\ldots,x_{k-1},x'_k,x_{k+1},\ldots,x_n)\\ &\quad - i_{M_1\times M_2\times\cdots\times M_n}(x_1,\ldots,x_{k-1},x''_k,x_{k+1},\ldots,x_n), \end{aligned}$$

or are of the form

$$\begin{aligned} &i_{M_1\times M_2\times\cdots\times M_n}(x_1,\ldots,x_{k-1},rx_k,x_{k+1},\ldots,x_n)\\ &\quad - r i_{M_1\times M_2\times\cdots\times M_n}(x_1,\ldots,x_{k-1},x_k,x_{k+1},\ldots,x_n), \end{aligned}$$

where $x_l, x'_l, x''_l \in M_l$ for $l = 1, 2, \ldots, n$, and $r \in R$. There is an $R$-multilinear function

$$j_{M_1\times M_2\times\cdots\times M_n}\colon M_1 \times M_2 \times \cdots \times M_n \to M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n,$$

where $j_{M_1\times M_2\times\cdots\times M_n}$ is the composition $\pi \circ i_{M_1\times M_2\times\cdots\times M_n}$ of the natural embedding

$$i_{M_1\times M_2\times\cdots\times M_n}\colon M_1 \times M_2 \times \cdots \times M_n \to F_R(M_1 \times M_2 \times \cdots \times M_n)$$

and the quotient homomorphism

$$\pi\colon F_R(M_1 \times M_2 \times \cdots \times M_n) \to M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n.$$

## 8.10 Tensor Products over Non-Commutative Rings

Let $R$ be a unital ring that is not necessarily commutative, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. These modules are Abelian groups under the operation of addition, and Abelian groups are modules over the ring $\mathbb{Z}$ of integers. We can therefore form their tensor product $M \otimes_{\mathbb{Z}} N$. This tensor product is an Abelian group.

Let $K$ be the subgroup of $M \otimes_{\mathbb{Z}} N$ generated by the elements

$$(xr) \otimes_{\mathbb{Z}} y - x \otimes_{\mathbb{Z}} (ry)$$

for all $x \in M$, $y \in N$ and $r \in R$, where $x \otimes_{\mathbb{Z}} y$ denotes the tensor product of $x$ and $y$ in the ring $M \otimes_{\mathbb{Z}} N$. We define the tensor product $M \otimes_R N$ of the right $R$-module $M$ and the left $R$-module $N$ over the ring $R$ to be the quotient group $M \otimes_{\mathbb{Z}} N/K$. Given $x \in M$ and $y \in N$, let $x \otimes y$ denote the image of $x \otimes_{\mathbb{Z}} y$ under the quotient homomorphism $\pi \colon M \otimes_{\mathbb{Z}} N \to M \otimes_R N$. Then

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$$

and

$$(xr) \otimes y = x \otimes (ry)$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$.

**Lemma 8.14** *Let $R$ be a unital ring, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. Then the tensor product $M \otimes_R N$ of $M$ and $N$ is an Abelian group that satisfies the following universal property:*

> *given any Abelian group $P$, and given any $\mathbb{Z}$-bilinear function $f \colon M \times N \to P$ which satisfies*
>
> $$f(xr, y) = f(x, ry)$$
>
> *for all $x \in M$, $y \in N$ and $r \in R$, there exists a unique Abelian group homomorphism $\varphi \colon M \otimes_R N \to P$ such that $f(x, y) = \varphi(x \otimes y)$ for all $x \in M$ and $y \in N$.*

## 8.11 Tensor Products of Bimodules

Let $Q$, $R$ and $S$ be unital rings, let $M$ be a $Q$-$R$-bimodule, and let $N$ be an $R$-$S$-bimodule. Then $M$ is a right $R$-module and $N$ is a left $R$-module. We can therefore form the tensor product $M \otimes_R N$ of $M$ and $N$ over the ring $R$. This tensor product is an Abelian group under the operation of addition.

Let $q \in Q$ and $r \in R$. The definition of bimodules ensures that $(qx)r = q(xr)$ for all $x \in M$. Let $L_q \colon M \times N \to M \otimes_R N$ be the function defined such that $L_q(x, y) = (qx) \otimes y$ for all $x \in M$ and $y \in N$. Then the function $f$ is $\mathbb{Z}$-bilinear. Moreover

$$L_q(xr, y) = (q(xr)) \otimes y = ((qx)r) \otimes y = (qx) \otimes (ry) = L_q(x, ry).$$

for all $x \in M$ and $y \in N$. It follows from Lemma 8.14 that there exists a group homomorphism $\lambda_q \colon M \otimes_R N \to M \otimes_R N$, where $\lambda_q(x \otimes y) = (qx) \otimes y$ for all $x \in M$ and $y \in N$. Similarly, given any element $s$ of the ring $S$, there exists a group homomorphism $\rho_s \colon M \otimes_R N \to M \otimes_R N$, where $\lambda_s(x \otimes y) = x \otimes (ys)$. We define $q\alpha = \lambda_q(\alpha)$ and $\alpha s = \rho_s(\alpha)$ for all $\alpha \in M \otimes_R N$. One can check that $M \otimes_R N$ is a $Q$-$S$-bimodule with respect to these operations of left multiplication by elements of $Q$ and right multiplication by elements of

$S$. Moreover, given any $Q$-$S$-bimodule $P$, and given any $Z$-bilinear function $f\colon M \times N \to P$ that satisfies

$$f(qx, y) = qf(x, y), \quad f(xr, y) = f(x, ry), \quad f(x, ys) = f(x, y)s$$

for all $x \in M$, $y \in N$, $q \in Q$, $r \in R$ and $s \in S$, there exists a unique $Q$-$S$ bimodule homomorphism $\varphi\colon M \otimes_R N \to P$ such that $f(x, y) = \varphi(x \otimes y)$ for all $x \in M$ and $y \in N$.

This constuction generalizes the definition and universal property of the tensor product of modules over a unital commutative ring $R$, in view of the fact that any module over a unital commutative ring $R$ may be regarded as an $R$-$R$-bimodule.

## 8.12 Tensor Products involving Free Modules

**Proposition 8.15** *Let $R$ and $S$ be unital rings, let $M$ be an $R$-$S$-bimodule and let $F_S X$ be a free left $S$-module on a set $X$. Then the tensor product $M \otimes_S F_S X$ is isomorphic, as an $R$-module, to $\Gamma(X, M)$, where $\Gamma(X, M)$ is the left $R$-module whose elements are represented as functions from $X$ to $M$ with only finitely many non-zero values, and where $(\lambda + \mu)(x) = \lambda(x) + \mu(x)$, and $(r\lambda)(x) = r\lambda(x)$ for all $\lambda, \mu \in \Gamma(X, M)$ and $r \in R$.*

**Proof** The elements of the free left $S$-module $F_S X$ are represented as functions from $X$ to $S$. Let $f\colon M \times F_S X \to \Gamma(X, M)$ be the $\mathbb{Z}$-bilinear function defined such that $f(m, \sigma)(x) = m\sigma(x)$ for all $m \in M$, $\sigma \in F_S X$ and $x \in X$. Then $f(ms, \sigma) = f(m, s\sigma)$ for all $m \in M$, $\sigma \in F_S X$ and $s \in S$. It follows from Lemma 8.14 that the function $f$ induces a unique homomorphism $\theta\colon M \otimes_S F_S X \to \Gamma(X, M)$ such that $\theta(m \otimes \sigma) = f(m, \sigma)$. Moreover $\theta$ is an $R$-module homomorphism.

Given $\mu \in \Gamma(X, M)$ we define

$$\varphi(\mu) = \sum_{x \in \operatorname{supp} \mu} \mu(x) \otimes \delta_x,$$

where $\operatorname{supp} \mu = \{x \in X : \mu(x) \neq 0\}$ and $\delta_x$ denotes the function from $X$ to $S$ which takes the value $1_S$ at $x$ and is zero elsewhere. Then $\varphi\colon \Gamma(X, M) \to M \otimes_S F_S X$ is also an $R$-module homomorphism. Now

$$\begin{aligned} \varphi(\theta(m \otimes \sigma)) &= \sum_{x \in \operatorname{supp} \sigma} m\sigma(x) \otimes \delta_x = \sum_{x \in \operatorname{supp} \sigma} m \otimes \sigma(x)\delta_x \\ &= m \otimes \left( \sum_{x \in \operatorname{supp} \sigma} \sigma(x)\delta_x \right) = m \otimes \sigma \end{aligned}$$

for all $m \in M$ and $\sigma \in F_S X$. It follows that $\varphi \circ \theta$ is the identity automorphism of the tensor product $M \otimes_S F_S X$.

Also

$$\theta(\varphi(\mu)) = \theta\left(\sum_{x \in \operatorname{supp} \mu} \mu(x) \otimes \delta_x\right) = \sum_{x \in \operatorname{supp} \mu} \theta(\mu(x) \otimes \delta_x)$$

for all $\mu \in \Gamma(X, M)$. But

$$\theta(\mu(x) \otimes \delta_x)(y) = \begin{cases} \mu(x) & \text{if } y = x; \\ 0 & \text{if } y \neq x. \end{cases}$$

It follows that

$$\theta(\varphi(\mu) = \sum_{x \in \operatorname{supp} \mu} \theta(\mu(x) \otimes \delta_x) = \mu$$

for all $\mu \in \Gamma(X, M)$. Thus $\theta \circ \varphi$ is the identity automorphism of $\Gamma(X, M)$. We conclude that $\theta: M \otimes_S F_S X \to \Gamma(X, M)$ is an isomorphism of $R$-modules, as required. ∎

Let $R$ be a unital ring. We can regard $R$ as an $R$-$\mathbb{Z}$-bimodule, where $rn$ is the sum of $n$ copies of $r$ and $r(-n) = -rn$ for all non-negative integers $n$ and elements $r$ of $R$. We may therefore form the tensor product $R \otimes_{\mathbb{Z}} A$ of the ring $R$ with any additive group $A$. (An additive group as an Abelian group where the group operation is expressed using additive notation.) This tensor product is an $R$-module. The following corollary is therefore a direct consequence of Proposition 8.15.

**Corollary 8.16** *Let $R$ be a unital ring, let $X$ be a set, and let $F_{\mathbb{Z}} X$ be the free Abelian group on the set $X$. Then $R \otimes_{\mathbb{Z}} F_{\mathbb{Z}} X \cong F_R X$. Thus the tensor product of the ring $R$ with any free Abelian group is a free $R$-module.*

## 8.13 The Relationship between Bimodules and Left Modules

Let $R$ and $S$ be unital rings with multiplicative identity elements $1_R$ and $1_S$, and let $S^{op}$ be the unital ring $(S, +, \overline{\times})$ whose elements are those of $S$, whose operation of addition is the same as that defined on $S$, and whose operation $\overline{\times}$ of multiplication is defined such that $s_1 \overline{\times} s_2 = s_2 s_1$ for all $s_1, s_2 \in S$.

We can then construct a ring $R \otimes_{\mathbb{Z}} S^{\text{op}}$. The elements of this ring belong to the tensor product of the rings $R$ and $S^{\text{op}}$ over the ring $\mathbb{Z}$ of integers, and the operation of addition on $R \otimes_{\mathbb{Z}} S^{\text{op}}$ is that defined on the tensor product. The operation of multiplication on $R \otimes_{\mathbb{Z}} S^{\text{op}}$ is then defined such that

$$(r_1 \otimes s_1) \times (r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 \overline{\times} s_2) = (r_1 r_2) \otimes (s_2 s_1).$$

**Lemma 8.17** *Let $R$ and $S$ be unital rings, and let $M$ be an $R$-$S$-bimodule. Then $M$ is a left module over the ring $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$, where*

$$(r_1 \otimes s_1) \times (r_2 \otimes s_2) = (r_1 r_2) \otimes (s_2 s_1)$$

*for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$, and where*

$$(r \otimes s).x = (rx)s = r(xs)$$

*for all $r \in R$, $s \in S$ and $x \in M$.*

**Proof** Given any element $x$ of $M$, let $b_x: R \times S \to M$ be the function defined such that $b_x(r, s) = (rx)s = r(xs)$ for all $r \in R$ and $s \in S$. Then the function $b_x$ is $\mathbb{Z}$-bilinear, and therefore induces a unique $\mathbb{Z}$-module homomorphism $\beta_x: R \otimes_{\mathbb{Z}} S^{\mathrm{op}} \to M$, where $\beta_x(r \otimes s) = b_x(r, s) = (rx)s$ for all $r \in R$, $s \in S$ and $x \in M$. We define $u.x = \beta_x(u)$ for all $u \in R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ and $x \in M$. Then $(u_1 + u_2).x = u_1.x + u_2.x$ for all $u_1, u_2 \in R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ and $x \in M$, because $\beta_x$ is a homomorphism of Abelian groups. Also $u.(x_1 + x_2) = u.x_1 + u.x_2$, because $b_{x_1+x_2} = b_{x_1} + b_{x_2}$ and therefore $\beta_{x_1+x_2} = \beta_{x_1} + \beta_{x_2}$.

Now

$$\begin{aligned}
(r_1 \otimes s_1).\left((r_2 \otimes s_2).x\right) &= (r_1 \otimes s_1).((r_2 x)s_2) = r_1(r_2(xs_2))s_1 \\
&= ((r_1 r_2)(xs_2))s_1 = (r_1 r_2)((xs_2)s_1) \\
&= (r_1 r_2)(x(s_2 s_1) = ((r_1 r_2) \otimes_{\mathbb{Z}} (s_2 s_1)).x \\
&= ((r_1 \otimes_{\mathbb{Z}} s_1) \times (r_2 \otimes_{\mathbb{Z}} s_2)).x
\end{aligned}$$

for all $r_1, r_2 \in R$, $s_1, s_2 \in S$ and $x \in M$. The bilinearity of the function $\beta_x$ then ensures that $u_1.(u_2.x) = (u_1 \times u_2).x$ for all $u_1, u_2 \in R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ and $x \in M$. Also $(1_R, 1_S).x = x$ for all $x \in M$, where $1_R$ and $1_S$ denote the identity elements of the rings $R$ and $S$. We conclude that $M$ is a left $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$, as required. ∎

Let $R$ and $S$ be unital rings, and let $M$ be a left module over the ring $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$. Then $M$ can be regarded as an $R$-$S$-bimodule, where $(rx)s = r(xs) = (r \otimes s).x$ for all $r \in R$, $s \in S$ and $x \in M$. We conclude therefore that all $R$-$S$-bimodules are left modules over the ring $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$, and vica versa. It follows that any general result concerning left modules over unital rings yields a corresponding result concerning bimodules.